Math 996 (Fall 2017): Quillen-K-Theory and Beyond

A Synopsis; Instructor: Satya Mandal, U. of Kansas

A graduate courses in Algebraic K-Theory, would usually mean a course in Classical K-theory. However, the Advent of Quillen's K-Theory [Q, 1972], may be considered as the end of Classical K-theory. What the K-Theorist of this age do, differ greatly, from the methods of the Classical K-theory.

1. Given a noetherian commutative ring A ($X = \mathrm{Spec}\,(A)$), or more generally any noetherian scheme X, in Classical K-Theory, there were attempts to define groups $K_i(X)$, for all integers $i \geq 0$. These groups $K_i(X)$ would be expected to fit into some long exact sequences.

2. Given an exact category $\mathscr{E}$, Quillen [Q], associated a topological space $\mathbf{K}(\mathscr{E})$ (a CW-complex). For $i \geq 0$, he defined $K_i(\mathscr{E}) = \pi_i\,(\mathbf{K}(\mathscr{E}))$, the i^{th}-homotopy group. Taking $\mathscr{E}$ as the category of projective A-modules, he defined $K_i(X) := K_i(\mathscr{E})$. This completed the quest for $K_i(X)$, as envisioned in (1).

Background, contents and goals: I plan to make this course accessible to any graduate student at KU. I will try to do it well enough, so that I may have a chance to publish the lecture notes.

1. The background needed in Category Theory will be covered.
   The background in Topological Homotopy theory (groups) would be covered.
   Combinatorial (equivalent) approach to Homotopy theory would be covered.

2. Basics of Quillen's K-Theory [Q] will be covered.

3. For a topological space X, there is a concept of Negative Homotopy groups $\pi_i(X)\ \forall\ i \leq -1$. The process is called de-looping. This leads to, for Exact categories $\mathscr{E}$, the definition of Negative K-groups, $K_i(\mathscr{E})\ \forall\ i \leq -1$. This will be covered.

4. The category of projective modules over a ring has a duality. Likewise, many exact categories $\mathscr{E}$ comes with a duality. For an exact category $\mathscr{E}$ with duality, there is another topological invariant called Grothedieck With Spaces. The theory is analogous to K-theory spaces, but more involved. We would touch on it, if time permits.

Emphasis: Emphasis would be on an overview. Technical details of proofs would be deemphasized. This approach would be essential mainly to save time. Further, technical details of proofs are sometimes not learnt in classes.

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