Math 291: Online Guidance **(Chalk Board)**

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Spring 2021

1 Frist Week

This is Linear algebra, Spring 21

Theorem For any system of linear equations there three possibilities:

1. The system has no solutions
2. The system has exactly one solutions.
3. The system has infinitely many solutions.

I also defined **equivalent** system.

Thursday

$$\begin{pmatrix} 1 & 11 & 0 & 6 \\ 4 & 1 & 0 & -3 \\ 11 & 0 & 1 & -7 \end{pmatrix}$$

Week Three: Thursday

Binary operations, Identity and Inverse

1. A binary operation is something that combines two object to produce a third object.

2. Let us consider the case that we combine same kind of objects X, Y and produce a new object of same kind $X \star Y$. Given such a binary operation, we can ask two questions

   (a) Is there an identity I?
   An object I is called identity, of this operation (product), if

   $$X \star I = I \star X = X$$

   We can ask, if there is an identity, is it unique?
   Answer: Yes
   Proof. Suppose J is another identity.

   $$J = J \star I = I$$

   Here the first equality holds because I is an identity, and
   the second equality holds because J is an identity.

   (b) Suppose there is an identity I. In this this case, given an object X, we ask does it have an inverse? This means, is there an object Y such that

   $$X \star Y = Y \star X = I$$

   In the case of square matrices of order n, I just proved,
   if there is an inverse of X, then there is only one (unique).
   This proof did not use any particular property of matrices.
   It only used, associativity property of multiplication.

Even and Odd Permutation

Suppose $\sigma = (i_1, i_2, \ldots, i_n)$ is a permutation of $1, 2, \ldots, n$.
If you get σ_1 frm σ, by switching two adjacent numbers, then σ_1 called a **transposition** of σ.
A σ is called **even** permutation, if even number of steps (transpositions) are needed to bring σ back to natural order of $1, 2, \ldots, n$. Otherwise σ is called an **Odd** permutation.

Given σ, either you need even number of permuations, or,
odd number of permuations, to bring it back to the natural order.
Define

$$sign(\sigma) = \begin{cases} 1 & If\ even \\ -1 & if\ odd \end{cases}$$

Week of 9 March

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(A) = ad - bc$$

Then,

$$Adj(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{and} \quad A^{-1} = \frac{1}{\det(A)} Adj(A)$$

Note,

$$AAdj(A) = Adj(A)A = \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} = \det(A) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

16 March 2021

1. Condition (3), (4), (5) means, V is a **Group** under addition.
2. Further, condition (2), means V is **commutative group**.

Recall
$$GL_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \text{A is invertible}\}$$
This is a groups under multiplication.

13 April 2021

Definition 1.1. A set X together with a **distance function**

$$d : X \times X \longrightarrow [0, \infty) \qquad \text{sending} \qquad (x, y) \mapsto d(x, y)$$

is called a **Metric space** if

1. $d(x, y) \geq 0$ for all $x, y \in X$.
2. $d(x, y) = 0 \Longleftrightarrow x = y$
3. $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. (**Triangular Inequality**): $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

15 April
Exam Problems:

Ex. 17

1. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is linearly independent, then it is a basis.
   So, we would be done in this case.

2. Suppose it is not linearly independent. So, they are linearly dependent.
   So, one of them is in the span of the rest.
   By relabeling, we can assume $\mathbf{v}_n \in Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}\}$.
   So,

   $$V = Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n\} = Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}\}$$

   Now, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}\}$ is linearly independent, the it is a basis.
   So, we are done.

3. We continue to repeat this process.
   And, the process must terminates, when the resulting subset is linearly independent. So, the resulting subset will be a basis.

Ex. 18

1. It is given that $V = Span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$.
   It follows from Ex. 17, that $n = dimV \leq r$.

2. By Theorem 4.5.2, it follows $k \leq \dim V = n$.

3. If $V = Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis.

4. If not, meaning

   $$V \neq Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \quad \text{then, there is} \quad \mathbf{v}_{k+1} \in V \ni \quad \mathbf{v}_{k+1} \notin Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

5. It follows,

   $$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k; \mathbf{v}_{k+1}\} \quad \text{is linealry independent}$$

   If

   $$V = Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k; \mathbf{v}_{k+1}\} \quad \text{then, it is a basis.}$$

6. Process must terminate, when total number of elements in enlarged set
   is $= n = \dim V$.

Ex 20:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Then.

$$|\lambda I_4 - A| = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & \lambda - a_{22} & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & \lambda - a_{33} & -a_{34} \\ -a_{41} & -a_{42} & -a_{43} & \lambda - a_{44} \end{vmatrix}$$

Now expand.