Chapter 7
§7.6 Complex Eigenvalues

Satya Mandal, KU

Complex Eigenvalues

- We continue to consider homogeneous linear systems with constant coefficients:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \quad \mathbf{A} \text{ is an } n \times n \text{ matrix with constant entries} \tag{1}$$

- In §7.5, we considered the situation when all the eigenvalues of **A**, were real and distinct. In this section, we consider when some of the eigen values are complex.
- As in §7.4, solutions of (1) will be denoted by

$$\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t).$$

Principle of superposition

- Recall the Principle of superposition and the converse (§7.4): IF $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are solution of (1), then, any constant linear combination

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)} \tag{2}$$

  is also a solution of the same system (1).
- The converse is also true, if Wronskian $W \neq 0$.
- Further, if r is an eigenvalue of $\mathbf{A}$ and ξ is an eigenvector for r then

$$\mathbf{x} = \xi e^{rt} \quad \text{is a solution of (1)} \tag{3}$$

Complex eigenvalues and vectors

- Suppose **A** has a complex eigenvalue $r_1 = \lambda + i\mu$ and $\xi^{(1)}$ is an eigenvector, for r_1. That means

$$(\mathbf{A} - (\lambda + i\mu)I)\xi^{(1)} = \mathbf{0}. \tag{4}$$

- Apply conjugation to (4): $(A - (\lambda - i\mu)I)\overline{\xi^{(1)}} = \mathbf{0}$. This means:
  - $r_2 = \overline{r_1} = \lambda - i\mu$ an eigenvalue of **A**.
  - And, $\xi^{(2)} = \overline{\xi^{(1)}}$ is an eigenvector of **A**, corresponding to r_2.

Continued: Two conjugate complex Solutions

- Two eigen values $r_1, r_2 = \overline{r_1}$ and the corresponding eigenvalues gives two solutions of (1):

$$\mathbf{x}^{(1)} = \xi^{(1)} e^{r_1 t}, \qquad \mathbf{x}^{(2)} = \xi^{(2)} e^{r_2 t} \tag{5}$$

- Write $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ real real vectors. Then,

$$\mathbf{x}^{(1)} = (\mathbf{a} + i\mathbf{b}) e^{(\lambda + i\mu)t} = (\mathbf{a} + i\mathbf{b})[e^{\lambda t}(\cos \mu t + i \sin \mu t)]$$

$$= e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) + i e^{\lambda t}(\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t)$$

Continued: Two Real Solutions

- Both real and imaginary part of $\mathbf{x}^{(1)}$ are solutions of (1), as follows:

$$\begin{cases} \mathbf{u} = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) \\ \mathbf{v} = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \end{cases} \qquad (6)$$

- These real solutions $\mathbf{u}, \mathbf{v}$ fit in very well as a part of a fundamental set of n solutions. There will be too many cases to make this statement precise. The textbook makes the statement in the next frame, where remaining eigenvalues are real and distinct.
- Often, we will consider systems of 2 or 3 equations. So, following statement will suffice, in most cases.

As part of Fundamental set

- Suppose $r_1 = \lambda + i\mu$, $r_1 = \lambda - i\mu$ are two conjugate eigenvalues of $\mathbf{A}$. As above, let $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$ is an eigenvector of r_1. Accordingly, the conjugate $\xi^{(2)} = \mathbf{a} + i\mathbf{b}$ is an eigenvector of r_2.
- Assume $r_3, \ldots, r_n$ be the remaining eigenvalues of $\mathbf{A}$. Let $\xi^{(i)}$ an eigenvector of r_i, for $i = 3, \ldots, n$.
- Further assume $r_3, \ldots, r_n$ are real and distinct.

Then, $\mathbf{u}, \mathbf{v}, \xi^{(3)}, \ldots, \xi^{(n)}$ forms a fundamental set of solutions of (1). Hence, any solution $\mathbf{x}$ has the form (2):

$$\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\xi^{(3)}e^{r_3 t} + \cdots + c_n\xi^{(n)}e^{r_n t} \tag{7}$$

Continued

- Above statement and the form of the general solution (7) hold in a much more general situation, without requiring $r_3, \ldots, r_n$ are real and distinct.
- It works, if we assume $\mathbf{u}, \mathbf{v}, \xi^{(3)}, \ldots, \xi^{(n)}$ are linearly independent. Which is equivalent to

$$\left| \begin{array}{cccccc} \mathbf{u} & \mathbf{v} & \xi^{(3)} & \cdots & \xi^{(n)} \end{array} \right| \neq 0$$

Sample I: Ex 5

Find the general solution (real valued) of the equation:

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x} \qquad (8)$$

- Eigenvalues of the coef. matrix **A**, are: given by

$$\begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = 0 \quad r = -1+i, -1-i$$

Eigenvectors

- Analytically, eigenvectors for $r = -1 + i$ is given by $(\mathbf{A} - rI)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The second row is $2+i$-times the first row. It follows:

$$\begin{pmatrix} 2-i & -1 \\ & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

With $x_1 = 1$, an eigenvector of $r = -1 + i$ is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The solution

- So, the real and the imaginary part of $\xi^{(1)}$ are:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

- With $r = -1 + i$, we have $\lambda = -1, \mu = 1$. By (6), the general solution of (8)

$$\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} = c_1 e^{-t}(\mathbf{a}\cos t - \mathbf{b}\sin t) + c_2 e^{-t}(\mathbf{a}\sin t + \mathbf{b}\cos t)$$

$$= c_1 e^{-t}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin t\right)$$

$$+ c_2 e^{-t}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos t\right)$$

Continued

- $\mathbf{x} =$

$$c_1 e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

Sample II: Ex 7

Find the general solution (real valued) of the equation:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x} \qquad (9)$$

- Eigenvalues of the coef. matrix **A**, are:

$$\begin{vmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{vmatrix} = 0$$

$$(1-r)\begin{vmatrix} 1-r & -2 \\ 2 & 1-r \end{vmatrix} = 0$$

So, $r = 1, 1 \pm 2i$

Eigenvectors

- Eigenvectors for $r = 1$ is given by $(\mathbf{A} - rI)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-1 & 0 & 0 \\ 2 & 1-1 & -2 \\ 3 & 2 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Use TI-84 (rref):

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1.5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

With $x_1 = 2$, an eigenvector of $r = 1$ is: $\xi^{(1)} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$.

The corresponding solution $\mathbf{x}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^{t}$

Eigenvectors

- Eigenvectors for $r = 1 + 2i$ is given by $(\mathbf{A} - rI)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-(1+2i) & 0 & 0 \\ 2 & 1-(1+2i) & -2 \\ 3 & 2 & 1-(1+2i) \end{pmatrix} \mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$\begin{cases} -2ix_1 = 0 \\ 2x_1 - 2ix_2 - 2x_3 = 0 \\ 3x_1 + 2x_2 - 2ix_3 = 0 \end{cases} \begin{cases} x_1 = 0 \\ -2ix_2 - 2x_3 = 0 \\ 2x_2 - 2ix_3 = 0 \end{cases} \begin{cases} x_1 = 0 \\ ix_2 + x_3 = 0 \\ 0 = 0 \end{cases}$$

With $x_3 = 1$, an eigenvector of $r = 1 + 2i$ is:

$$\xi^{(2)} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i$$

Solutions corresponding to $r = 1 \pm 2i$

By (6) two real solutions, corresponding to $r = 1 \pm 2i$ are:

$$\begin{cases} \mathbf{u} = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) \\ \mathbf{v} = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \end{cases}$$

$$\begin{cases} \mathbf{u} = e^{t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\sin 2t\right) = e^{t}\begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} \\ \mathbf{v} = e^{t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cos 2t\right) = e^{t}\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \end{cases}$$

The general solution

Combining $\mathbf{x}^{(1)}, \mathbf{u}, \mathbf{v}$, by (7), the general solution of (9) is

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{u} + c_3\mathbf{v}$$

$$= c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t + c_2 e^t \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix}$$

Sample III: Ex 9

Solve the initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

- Eigenvalues of the coef. matrix **A**, are: given by

$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = 0 \quad r = -1+i, -1-i$$

Eigenvectors

- Analytically, eigenvectors for $r = -1 + i$ is given by $(\mathbf{A} - rI)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The first row is $2-i$-times the second row. It follows:

$$\begin{pmatrix} 0 & 0 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- With $x_2 = 1$, an eigenvector of $r = -1 + i$ is

$$\xi^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- So, the real and the imaginary part of $\xi^{(1)}$ are:

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The solution

- With $r = -1 + i$, we have $\lambda = -1, \mu = 1$. By (6), the general solution of (8)

$$\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} = c_1 e^{-t}(\mathbf{a}\cos t - \mathbf{b}\sin t) + c_2 e^{-t}(\mathbf{a}\sin t + \mathbf{b}\cos t)$$

$$= c_1 e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right)$$

$$+ c_2 e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t\right)$$

Continued

- $\mathbf{x} =$

$$c_1 e^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

- Use the initial value condition:

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow c_1 = 1, c_2 = -1$$

The Answer

- So, the answer is:

$$\mathbf{x} = e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$$

§7.6 Assignments and Homework

- Read Example 1, 3 (They are helpful).
- Homework: §7.6 See Homework Site!

Ex 5

In what follows, I gave Matlab output. You can ignore the rest.

$$A = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$$

$r = -1 \pm i$

$$V = \begin{pmatrix} 0.3651 + 0.1826i & 0.3651 - 0.1826i \\ 0.9129 & 0.9129 \end{pmatrix}$$

Ex 6

$$A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$$

$r = \pm 3i$

$$\begin{pmatrix} -0.1690 - 0.5071i & -0.1690 + 0.5071i \\ 0.8452 & 0.8452 \end{pmatrix}$$

Ex 7

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$$

$$r = 1.0000 \pm 2.0000i, 1$$

$$V = \begin{pmatrix} 0 & 0 & 0.4851 \\ 0.7071 & 0.7071 & -0.7276 \\ 0 - 0.7071i & 0 + 0.7071i & 0.4851 \end{pmatrix}$$

Ex 8

$$A = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}$$

$$r = -1.0000 \pm 1.4142i, -2$$

$$V = \begin{pmatrix} -0.4714 + 0.3333i & -0.4714 - 0.3333i & 0.6667 \\ 0.2357 + 0.3333i & 0.2357 - 0.3333i & -0.6667 \\ -0.7071 & -0.7071 & 0.3333 \end{pmatrix}$$

Ex 9

$$A = A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$$

$r = -1 \pm i$

$$V = A = \begin{pmatrix} 0.9129 & 0.9129 \\ 0.3651 - 0.1826i & 0.3651 + 0.1826i \end{pmatrix}$$

Ex 10

$$A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$$

$r = -2 \pm i$

$$V = \begin{pmatrix} 0.8165 & 0.8165 \\ 0.4082 + 0.4082i & 0.4082 - 0.4082i \end{pmatrix}$$