Determinant of a Matrix

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Goals

- We will define determinant of SQUARE matrices, inductively, using the definition of Minors and cofactors.
- We will see that determinant of triangular matrices is the product of its diagonal elements.
- Determinants are useful to compute the inverse of a matrix and solve linear systems of equations (Cramer's rule).

Overview of the definition

- Given a square matrix A, the determinant of A will be defined as a scalar, to be denoted by $det(A)$ or $|A|$.
- We define determinant inductively. That means, we first define determinant of 1×1 and 2×2 matrices. Use this to define determinant of 3×3 matrices. Then, use this to define determinant of 4×4 matrices and so.

Determinant of 1×1 and 2×2 matrices

- For a 1×1 matrix $A = [a]$ define $det(A) = |A| = a$.
- Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad define \quad det(A) = |A| = ad - bc.$$

Example 1

Let

$$A = \begin{pmatrix} 2 & 17 \\ 3 & -2 \end{pmatrix} \quad then \quad det(A) = |A| = 2*(-2)-17*3 = -53$$

Example 2

Let

$$A = \begin{pmatrix} 3 & 27 \\ 1 & 9 \end{pmatrix} \quad then \quad det(A) = |A| = 3 * 9 - 1 * 27 = 0.$$

Minors of 3×3 matrices

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then, the **Minor** M_{ij} of a_{ij} is defined to be the determinant of the 2×2 matrix obtained by deleting the i^{th} row and j^{th} column.

For example

$$M_{22} = \begin{vmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Like wise

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

Cofactors of 3×3 matrices

Let A the 3×3 matrix as in the above frame. Then, the **Cofactor** C_{ij} of a_{ij} is defined, by some sign adjustment of the minors, as follows:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, using the above frame

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = a_{22}a_{33} - a_{23}a_{33}$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -(a_{11}a_{32} - a_{12}a_{31})$$

$$C_{32} = (-1)^{3+2} M_{32} = -(a_{11}a_{23} - a_{13}a_{21}).$$

Determinant of 3×3 matrices

Let A be the 3×3 matrix as above. Then the **determinant** of A is defined by

$$\det(A) = |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This definition may be called "definition by expansion by cofactors, along the first row". It is possible to define the same by expansion by second of third row, which we will be discussed later.

Example 3

Let

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 0 \\ -2 & 1 & 1 \end{vmatrix}$$

Compute the minor M_{11}, M_{12}, M_{13}, the cofactors C_{11}, C_{12}, C_{13} and the determinant of A.

Solution:

Then minors

$$M_{11} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}, M_{12} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix}, M_{13} = \begin{vmatrix} 3 & -2 \\ -2 & 1 \end{vmatrix}$$

Or

$$M_{11} = -2, \quad M_{12} = 3, \quad M_{13} = -1$$

Continued

So, the cofactors

$$C_{11} = (-1)^{1+1} M_{11} = -2, \quad C_{12} = (-1)^{1+2} M_{12} = -3,$$

$$C_{13} = (-1)^{1+3} M_{13} = -1$$

So,

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2*(-2) + 1*(-3) + 1*(-1) = -8$$

Inductive process of definition

- We defined determinant of size 3×3, using the determinant of 2×2 matrices.
- Now we can do the same for 4×4 matrices. This means first define minors, which would be determinant of 3×3 matrices. Then, define Cofactors by adjusting the sign of the Minors.Then, use the cofactors fo define the determiant of the 4×4 matrix.
- Then, we can define minors, cofactors and determinant of 5×5 matrices. The process continues.

Minors of $n \times n$ Matrices

We assume that we know how to define determinant of $(n-1) \times (n-1)$ matrices. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

be a square matrix of size $n \times n$. The **minor** M_{ij} of a_{ij} is defined to be the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column.

Cofactors and Detarminant of $n \times n$ Matrices

Let A be a $n \times n$ matrix.

- Define

$$C_{ij} = (-1)^{i+j} M_{ij} \quad \textit{which iscalled the } \textbf{cofactor} \textit{ of } a_{ij}.$$

- Define

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

This would be called a definiton by expasion by cofactors, along first row.

Alternative Method for 3×3 matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Form a new 3×5 matrix by adding first and second column to A:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & 31 & a_{32} \end{matrix}$$

Continued

Then $|A|$ can be computed as follows:

- add the product of all three entries in the three left to right diagonals.
- add the product of all three entries in the three right to left diagonals.
- Then, $|A|$ is the difference.

Definition.

Definitions. Let A be a $n \times n$ matrix.

- We say A is Upper Triangular matrix, all entries of A below the main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i > j$.
- We say A is Lower Triangular matrix, all entries of A above he main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i < j$.

Theorem

Theorem Let A be a triangular matrix of order n. Then $|A|$ is product of the main-diagonal entries. Notationally,

$$|A| = a_{11}a_{22}\cdots a_{nn}.$$

Proof. The proof is easy when $n = 1, 2$. We prove it when $n = 3$. Let use assume A is lower triangular. So,

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Continued

We expand by the first row:

$$|A| = a_{11}C_{11} + 0C_{12} + 0C_{13} = a_{11}C_{11}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

For upper triangular matrices, we can prove similarly, by column expansion. For higher order matrices, we can use mathematical induction. ■

Example

Compute the determiant, by expansion by cofactors, of

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

Solution.

- The cofactors

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 4 \\ 0 & 2 \end{vmatrix} = 8, C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = 2$$

- $$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} = -4$$
- So, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} =$
$$2 * 8 + (-1) * 2 + 3 * (-4) = 2$$

Example

$$Let \quad A = \begin{pmatrix} 3 & 7 & -3 & 13 \\ 0 & -7 & 2 & 17 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix} \quad Compute\ det(A).$$

Solution. This is an upper triangular matrix. So, $|A|$ is the product of the diagonal entries. So

$$|A| = 3 * (-7) * 4 * 5 = -420.$$

Example

$$Solve \quad \begin{vmatrix} x+3 & 1 \\ -4 & x-1 \end{vmatrix} = 0$$

Solution. So,

$$(x+3)(x-1) - 1 * (-4) = 0 \quad or \quad x^2 + 2x + 1 = 0$$

$$(x+1)^2 = 0 \quad or \quad x = -1.$$