# §7.3 System of Linear (algebraic) Equations
# Eigen Values, Eigen Vectors

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# Systems of Linear Equations

Consider a system of $m$ linear equations, in $n$ (unknown) varibales:

$$\begin{array}{llllll} a_{11}x_1+ & a_{12}x_2+ & a_{13}x_3+ & \cdots+ & a_{1n}x_n & = b_1 \\ a_{21}x_1+ & a_{22}x_2+ & a_{13}x_3+ & \cdots+ & a_{2n}x_n & = b_2 \\ a_{31}x_1+ & a_{32}x_2+ & a_{33}x_3+ & \cdots+ & a_{3n}x_n & = b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}x_1+ & a_{m2}x_2+ & a_{m3}x_3+ & \cdots+ & a_{mn}x_n & = b_m \end{array} \quad (1)$$

where $a_{ij}, b_j$ are real or complex numbers.

# Continued

- Write

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

  Then, $\mathbf{A}$ is called the coefficient matrix of the system (1). We also write $\mathbf{A} = (a_{ij})$.
- In matrix form, the system (1) is written as

$$\mathbf{Ax} = \mathbf{b} \tag{2}$$

# The Homogeneous Equation

- If $\mathbf{b} = \mathbf{0}$, then the system (2) would be called a homogeneous system. So,

$$\mathbf{Ax} = \mathbf{0} \tag{3}$$

is a homogeneous system of linear equation.

- Then, $\mathbf{x} = \mathbf{0}$ is a solution of the homogeneous system (3), to be called the trivial solution.

# A system and the homogeneous system

- Suppose $\mathbf{x}^{(0)}$ is a solution of the system (2): $\mathbf{Ax} = \mathbf{b}$.
- Then, any solution of (2): $\mathbf{Ax} = \mathbf{b}$ is of the form

$$\mathbf{x} = \mathbf{x}^{(0)} + \xi \tag{4}$$

where $\xi$ is a solution of the corresponding homogeneous system $\mathbf{Ax} = \mathbf{0}$.

# Augmented Matrix

- Corresponding to system (1), define the augmented matrix

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \qquad (5)$$

- In deed, the system (1) and the augmented matrix (5) has the same information/data. The Up-shot: the row operations performed on system (1), can be performed on the augmented matrix (5), in stead.

# Solving the system (1)

- There are three possibilities:
  - The system (1), may not have any solution.
  - The system (1), may have infinitely many solution.
  - The system (1), may have a unique solution. For this possibility, we need at least $n$ equations.
- To solve system (1), we can use TI-84 (ref, rref). Cosult any TI-84 site for instructions.

# $n = m$: System of $n$ equations and $n$ unknown

The textbook focuses on the case when $m = n$: the number of equations is same as number of unknown $x_1, \ldots, x_n$. In this section we assume $n = m$

- When $n = m$, then the coefficient matrix $\mathbf{A}$ of (1) is a square matrix of size $n \times n$.
- Recall, a square matrix $\mathbf{A}$ is invertible $\iff |\mathbf{A}| \neq 0$.
- If $|A| \neq 0$, then the unique solution of system (2)

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{is} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{6}$$

# Linear Indpendence

- A set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of vectors (in $\mathbb{R}^n$) is said to be linearly dependent over $\mathbb{R}$ if there are scalars $c_1, \ldots, c_k$ in $\mathbb{R}$, not all zero such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$.
- Likewise, a set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of vectors (in $\mathbb{C}^n$) is said to be linearly dependent over $\mathbb{C}$ if there are scalars $c_1, \ldots, c_k$ in $\mathbb{C}$, not all zero such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$.
- A set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of vectors is said to be linearly independent over $\mathbb{R}$ or $\mathbb{C}$, if they are not linearly dependent. That means, if

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0} \implies c_1 = c_2 = \cdots = c_k = 0.$$

# Continued

- Given a set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$) of vectors, we can form an $n \times k$ matrix $\mathbf{X} := \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_k \end{pmatrix}$.
- Then, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly independent, if $\mathbf{Xc} = \mathbf{0} \Longrightarrow \mathbf{c} = \mathbf{0}$. In other words, $\mathbf{Xc} = \mathbf{0}$ has no non-trivial solution.
- For $n$ such vectors, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$), they are linearly independent, if the determinant $|\mathbf{X}| \neq 0$.

# Eigenvalues and Eigenvectors

Suppose $\mathbf{A}$ is a square matrix of size $n \times n$.

- A scalar $\lambda \in \mathbb{C}$ is said to be an Eigenvalue of $\mathbf{A}$, if $|\mathbf{A} - \lambda\mathbf{I}| = 0$.
- The following are equivalent:
  - $\lambda \in \mathbb{C}$ is an Eigenvalue of $\mathbf{A}$
  - $|\mathbf{A} - \lambda\mathbf{I}| = 0$
  - The system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ has nontrivial solutions.
  - There are non-zero vectors $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$.
- Accordingly, a vector $\mathbf{x} \neq \mathbf{0}$ is said to be an eigenvector, for an eigenvalue $\lambda$ of $\mathbf{A}$, if $\mathbf{Ax} = \lambda\mathbf{x}$.

# Continued

- Eigenvalues are also called characteristic roots of **A**. (*The german word "eigen" means "particular" or "peculier".*)
- The equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$, is a polynomial equation in $\lambda$, of degree $n$, to be called the characteristic equation of **A**.
- Counting multiplicity of roots, the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$, has $n$ complex roots.
- Matlab can be used to compute eigenvalues and eigenvectors. Consult instructions in my site. The commands eig(A), [V,D]=eig(A) will be useful. However, Matlab does not work too well in this case. Eventually, we will use TI-84 to handle all these. Although, TI-84 does not have any direct command to do all these.

- Sometimes, there is no choice but to use analytic methods. This will be the case, when we have to deal with complex eigenvalues.
- Main thrust of this section is to compute eigenvalues and eigenvectors.

# Sample I: Ex 17

Find the eigenvalues and the corresponding eigenvector of

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \text{Use Matlab} \quad eig[V, D]$$

- Analytically: The characteristic equation:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) + 8 = 0 \Longleftrightarrow \lambda^2 - 2\lambda + 5 = 0$$

$$\text{Eigenvalues are} \qquad \lambda = 1 \pm 2i$$

# Eigenvectors for $\lambda = 1 + 2i$

To compute an eigenvector $\lambda = 1 + 2i$, we solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (2-2i)x_1 - 2x_2 = 0 \\ 4x_1 - (2+2i)x_2 = 0 \end{cases} \Longrightarrow \begin{cases} (1-i)x_1 - x_2 = 0 \\ 2x_1 - (1+i)x_2 = 0 \end{cases}$$

# Continued

Subtracting $1+i$-times the first equation from the second, we get

$$\left\{ \begin{array}{c} (1-i)x_1 - x_2 = 0 \\ 0 = 0 \end{array} \right. \implies \left\{ \begin{array}{c} (1-i)x_1 - x_2 = 0 \\ x_2 = (1-i)x_1 \end{array} \right.$$

Taking $x_1 = 1$, an eigenvector for $\lambda = 1 + 2i$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \tag{7}$$

# Eigenvectors for $\lambda = 1 - 2i$

- An eigenvectors for $\lambda = 1 - 2i$ can be computed, as in the case of its conjugate $1 + 2i$.
- Alternately, An eigenvectors for $\lambda = 1 - 2i$ is the conjugate of (7):

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}$$

# Sample II: Ex 20

Find the eigenvalues and the corresponding eigenvector of

$$\mathbf{A} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \qquad \text{Use Matlab} \quad eig[V, D]$$

- The characteristic equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 3 = 0 \iff \lambda^2 - 4 = 0$$

$$\text{Eigenvalues are} \qquad \lambda = 2, -2$$

# Eigenvectors for $\lambda = 2$

For $\lambda = 2$, solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-2 & \sqrt{3} \\ \sqrt{3} & -1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + \sqrt{3}x_2 = 0 \\ \sqrt{3}x_1 - 3x_2 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = \sqrt{3}x_2 \\ 0 = 0 \end{cases}$$

*The $2^{nd}$-line is obtained by adding $\sqrt{3}$-times the first equation to the second.*

# Continued

Taking $x_2 = 1$, an eigenvector for $\lambda = 2$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{8}$$

- Since $\lambda = 2$ has multiplicity one, we expect only one linearly independent eigenvector for $\lambda = 2$.

# Eigenvectors for $\lambda = -2$

For $\lambda = -2$, solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1+2 & \sqrt{3} \\ \sqrt{3} & -1+2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{array}{c} 3x_1 + \sqrt{3}x_2 = 0 \\ \sqrt{3}x_1 + x_2 = 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{c} 0 = 0 \\ x_2 = -\sqrt{3}x_1 \end{array} \right.$$

*The $1^{st}$-line is obtained by subtracting $\sqrt{3}$-times the scond equation to the first.*

# Continued

Taking $x_1 = 1$, an eigenvector for $\lambda = -2$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{9}$$

- Since $\lambda = -2$ has multiplicity one, we expect only one linearly independent eigenvector for $\lambda = -2$.

# Sample III: Ex 23

Find the eigenvalues and the corresponding eigenvector of

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}. \qquad \text{Use Matlab} \quad eig[V, D]$$

- Analytically: The characteristic equation:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix} = 0$$

# Continued

$$(3-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -4 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ -2 & -1-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 4-\lambda \\ -2 & -4 \end{vmatrix} = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \Longrightarrow$$

$$-\lambda^2(\lambda-1)+5\lambda(\lambda-1)-6(\lambda-1) = -(\lambda-1)(\lambda^2-5\lambda+6) = 0 \Longrightarrow$$

$$-(\lambda-1)(\lambda-2)(\lambda-3) = 0 \Longrightarrow \lambda = 1, 2, 3$$

are the eigenvalues of **A**.

# Eigenvectors for $\lambda = 1$

For $\lambda = 1$, solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 3-1 & 2 & 2 \\ 1 & 4-1 & 1 \\ -2 & -4 & -1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (10)$$

$$\left\{ \begin{array}{c} 2x_1 + 2x_2 + 2x_3 = 0 \\ x_1 + 3x_2 + x_3 = 0 \\ -2x_1 - 4x_2 - 2x_3 = 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{c} x_1 + x_2 + x_3 = 0 \\ x_1 + 3x_2 + x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{array} \right.$$

# Continued

Subtracting first equation from second and third:

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_2 = 0 \\ x_2 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = -x_3 \\ x_2 = 0 \\ x_2 = 0 \end{cases}$$

Taking $x_3 = 1$, an eigenvector for $\lambda = 1$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{11}$$

- Since $\lambda = 1$ has multiplicity one, we expect only one linearly independent eigenvector for $\lambda = 1$.
- It would be much simpler, if we use TI-84 (rref) to solve (10).

# Eigenvectors for $\lambda = 2$

For $\lambda = 2$, solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 3-2 & 2 & 2 \\ 1 & 4-2 & 1 \\ -2 & -4 & -1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

Use rref in TI-84:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

# Continued

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \\ 0 = 0 \end{cases} \implies \begin{cases} x_1 = -2x_2 \\ x_3 = 0 \\ 0 = 0 \end{cases}$$

Taking $x_1 = 1$, an eigenvector for $\lambda = 2$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

# Eigenvectors for $\lambda = 3$

For $\lambda = 3$, solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 3-3 & 2 & 2 \\ 1 & 4-3 & 1 \\ -2 & -4 & -1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

Use rref in TI-84:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow$$

# Continued

$$\left\{\begin{array}{c} x_1 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array}\right. \implies \left\{\begin{array}{c} x_1 = 0 \\ x_2 = -x_3 \\ 0 = 0 \end{array}\right.$$

Taking $x_3 = 1$, an eigenvector for $\lambda = 3$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

# §7.3 Assignments and Homework

- Read Example 4-5 (They are helpful).
- Homework: §7.3 Se the Homework Site!