Chapter 2: First Order DE
§2.4 Linear vs. Nonlinear DEs

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First Order DE

- We recall the general form of the First Oreder DEs (FODE):
$$\frac{dy}{dt} = f(t, y) \tag{1}$$
where $f(t, y)$ is a function of both the dependent variable t and the (unknown) dependent variable y.
- A DE, with an initial value condition $y(t_0) = y_0$ is called an Initial Value Problem (IVP).
- In §2.1, we worked with Linear FODEs. In this section (§2.4), we compare Linear vs. Nonlinear FODE.

Linear vs. Nonlinear DE

- A 1^{st}-order DE is called Linear, if it has the form

$$y' + p(t)y = g(t) \tag{2}$$

  Else, it is called nonlinear.
- Given a DE (linear or not), we ask:
  - Does all initial value problems (IVP) have a solution y?
  - If there is a solution $y = y(t)$, what is its domain?
  - When an IVP has a solution, is it unique? In other words, could an IVP have more than one solution?
- For linear IVP, it is easier to answer these questions.

Existance and Uniqueness of solutions

Theorem 2.4.1: Consider the 1^{st}-order linear IVP

$$\begin{cases} y' + p(t)y & = g(t) \\ y(t_0) & = y_0 \end{cases} \tag{3}$$

Assume $p(t), g(t)$ are continuous on an interval $I : \alpha < t < \beta$ and t_0 is in I. Then,

- The IVP (3) has a solution $y = \varphi(t)$.
- The domain of $y = \varphi(t)$ is I.
- The solution $y = \varphi(t)$ is unique, on I.

Continued

- The proof of the existance is done by defining integrating factor $\mu(t) = \exp \int p(t)dt$, as we solved problems in §2.1.
- For nonlinear FO-IVP $y' = f(t, y), y(t_0) = y_0$:
  - In general, the solutions of such DEs need not be unique.
  - But theorem 2.4.1 would be true in a more restricted sense (see theorem 2.4.2).

Exercise 4, pp 76

Consider the initial value problem (IVP)

$$\begin{cases} (4-t^2)y' + 2ty & = 3t^2 \\ y(-3) & = 1 \end{cases}$$

Determine the (without solving) an interval in which this IVP has a unique solution.

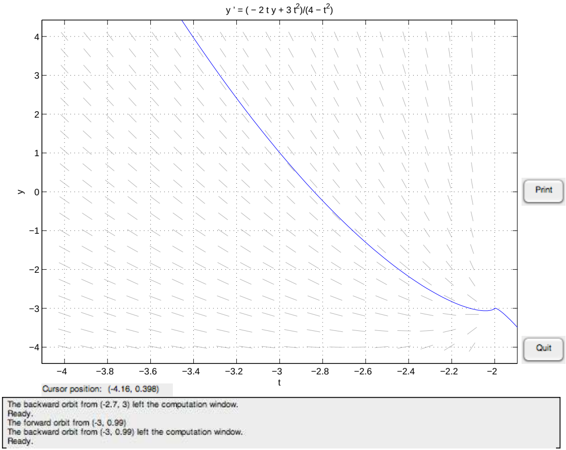
- Write the equation in the form (3), as in theorem 2.4.1:

$$\begin{cases} y' + \frac{2t}{4-t^2}y & = \frac{3t^2}{4-t^2} \\ y(-3) & = 1 \end{cases}$$

Continued

- $p(t) = \frac{2t}{4-t^2}$ is not defined at $t = -2, 2$. Split the number line as: $(-\infty, -2), (-2, 2), (2, \infty)$.
- Both $p(t) = \frac{2t}{4-t^2}$, $g(t) = \frac{3t^2}{4-t^2}$ are continuous on the intervals $(-\infty, -2), (-2, 2), (2, \infty)$.
- The initial t-value $t = -3$ is in $(-\infty, -2)$
- By theorem 2.4.1 the IVP has a unique solution on $(-\infty, -2)$.

The Direction Field: There should be a vertical tangent at $x = -2$, which is not clear form the dfield.



Exercise 3, pp 76

Consider the initial value problem (IVP)

$$\begin{cases} y' + (\tan t)y & = \sin t \\ y(\pi) & = 0 \end{cases}$$

Determine the (without solving) an interval in which this IVP has a unique solution.

- The equation in the form (3), as in theorem 2.4.1.
- $\tan t$ is not defined at $t = \pm\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.

Continued

- Accordingly, split the number line as:

$$\cdots, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \cdots$$

- Both $p(t) = \tan t$, $g(t) = \sin t$ are continuous on these intervals.
- The initial t-value $t = \pi$ is in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
- By theorem 2.4.1 the IVP has a unique solution on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Nonlinear DE

- Nonlinear DEs would not behave as nicely as in theorem 4.2.1.
- Uniqueness is not guaranteed.
- Solutions, if exist, may come out in an implicit form.
- Some DEs may not have an analytic solution. In such cases, numerical solutions would be an option.

Assignments and Homework

- Read Example 1, 2 (§2.4)
- Suggested Problems: Exercise §2.4 1-6 (page 76)
- Homework: §2.4 See Homework Site!