Chapter 2: First Order DE
§2.8 Numerical Solutions: Euler's Method

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First Order DE

- Recall the general form of the First Order ODEs (FODE):

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

- We can give analytic solutions to an ODE (1), only when it has some particular structure
(e.g. Linear, separable, Homogeneous, Bernoulli's, Exact and others).

Objective

- For a solution $y = \varphi(t)$ of (1), passing through (t_0, y_0), where $y_0 := \varphi(t_0)$), we have the following:
  - The tangent to the graph of $y = \varphi(t)$, at (t_0, y_0), is $m_0 = f(t_0, y_0)$. Hence, the equation to the tangent is

    $$y - y_0 = m_0(t - t_0), \quad \text{which can be computes from (1),}$$

    without actually computing $y = \varphi(t)$.
  - It also appears that we can sketch the graph of $y = \varphi(t)$, approximately, just by connecting the direction fields.
- In this section, we compute approximate solutions to the ODE (1), following the above.

Euler's Method

Let $y = \varphi(t)$ be a solution to the ODE (1), passing through a point (t_0, y_0), (hence $y_0 = \varphi(t_0)$).

- Rewrite the equation to the tangent to $y = \varphi(t)$,

$$\text{at} \quad (t_0, y_0): \qquad y = y_0 + f(t_0, y_0)(t - t_0)$$

- **The Notation** "$\approx$" would mean "approximately equal".
- If $t = t_1$ is close enough to t_0 then $\varphi(t_1) \approx y_0 + f(t_0, y_0)(t_1 - t_0)$. So, use

$$y_1 := y_0 + f(t_0, y_0)(t_1 - t_0) \quad \text{as an approximation to } \varphi(t_1).$$

Continued: Euler's Method

- Compare three lines:

$$\begin{cases} y = \varphi(t_1) + f(t_1, \varphi(t_1))(t - t_1) \\ y = y_1 + f(t_1, \varphi(t_1))(t - t_1) \\ y = y_1 + f(t_1, y_1)(t - t_1) \end{cases}$$

  The first line is the tangent to $y = \varphi(t)$, at $(t_1, \varphi(t_1))$. The 2^{nd}-line is parallel to the 1^{st}, passing through (t_1, y_1). The 3^{rd} passes through (t_1, y_1), with slope $= f(t_1, y_1)$.
- Since $y_1 \approx \varphi(t_1)$, use the 3^{rd}-line as an approximation to the first, if t is close enough to t_1.

- It $t = t_2$ is close enough to t_1, then

  $$\varphi(t_2) \approx \varphi(t_1) + f(t_1, \varphi(t_1))(t_2 - t_1) \approx y_1 + f(t_1, y_1)(t_2 - t_1)$$

  Use

  $$y_2 := y_1 + f(t_1, y_1)(t_2 - t_1) \quad \text{as an approximation to } \varphi(t_2).$$

- The process continues, and we have a sequence of points

  $$(t_0, y_0), (t_1, y_1), (t_2, y_2), \cdots, (t_n, y_n), \cdots$$

  with $\varphi(t_n) \approx y_n$.

Problem solving: Euler's Method

- Given an initial value problem (IVP)

$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases} \tag{2}$$

we will be asked to use Euler Method and approximate $\varphi(T)$, for some T.

  - Startin at $t = t_0$, attempt to reach T, in n equal jump of time interval h.
  - Either h or n will be given. We will have $h = \frac{T - t_0}{n}$.
  - We will take $t_0 = t_0, t_1 = t_0 + h, t_2 = t_1 + h, \ldots$.
  - We will have $\varphi(t_n) \approx y_n = y_{n-1} + f(t_{n-1}, y_{n-1})h$

Tools: Matlab and Excel

- A word of wisdom: Never do any computation by hand.
- For this section, use one or both of the following:
  - Use MS excel
  - Use my matlab program "Euler14". Direction is given in my site.
  - To use "Euler14" give command $Euler14(n, t_0, t_1, y_0)$, where (t_0, y_0) is the initial value, t_1 is the final t-value. And $n = \frac{t_1 - t_0}{h}$.

Example 1

Consider the IVP

$$\begin{cases} \frac{dy}{dt} = 2t \\ y(0) = 1 \end{cases}$$

- Compute the analytic solution $y = \varphi(t)$ of the ODE, and evaluate $\varphi(1)$.
- Use Euler's Method to approximate the solution at $t = 1$ with $h = .1, .05, .025$
- Compare that actual value $\varphi(1)$ and the approximated value.

Solution:

The ODE can be solved by a simple antiderivative:

$$y = \varphi(t) = \int 2tdt + c = t^2 + c \Longrightarrow y = \varphi(t) = t^2 + 1$$

So, $\varphi(1) = 1$.
Next, use Euler Method Approximation. We give two options:

- Use simple excel program.
- Use the Matlab program Euler14 that I will give you.

Euler Method Approximation

We have

$$y_n = y_{n-1} + f(t_{n-1}, y_{n-1})h = y_{n-1} + 2t_{n-1}h$$

We do some of them by hand: We have, with $h = .1$:

- $t_0 = 0$ anr $y_0 = 1$.
- $t_1 = .1$ and $y_1 = 1 + 2 * 0 * .1 = 1$
- $t_2 = t_1 + h = .2$ and $y_2 = 1 + 2 * .1 * .1 = 1.02$
- $t_3 = t_2 + h = .3$ and $y_3 = 1.02 + 2 * .2 * .1 = 1.06$
- $t_4 = t_3 + h = .4$ and$y_4 = 1.06 + (2 * .3) * .1 = 1.12$

Continued

For this first problem, we do a chart with the actual values (with $h = .1$)

t_i	y_i (Approximation)	Actual $\varphi(t) = t^2 + 1$
0	1	1
.1	1	1.01
.2	1.02	1.04
.3	1.06	1.09
.4	1.12	1.16
...	...	...
1	1.9	2

Euler14 Outputs

With $h = .1$

t_i	y_i
0	1.0000
0.1000	1.0000
0.2000	1.0200
0.3000	1.0600
0.4000	1.1200
0.5000	1.2000
0.6000	1.3000
0.7000	1.4200
0.8000	1.5600
0.9000	1.7200
1.0000	1.9000

Euler14 Outputs

With $h = .05$

t_i	y_i
0	1.0000
0.0500	1.0000
0.1000	1.0050
0.1500	1.0150
0.2000	1.0300
...	...
0.8000	1.6000
0.8500	1.6800
0.9000	1.7650
0.9500	1.8550
1.0000	1.9500

21 lines.

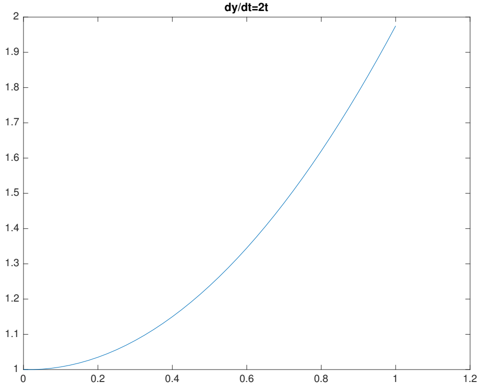
Euler14 Outputs

With $h = .025$

t_i	y_i
0	1.0000
0.0250	1.0000
0.0500	1.0012
0.0750	1.0037
$\cdots$	$\cdots$
0.9000	1.7875
0.9250	1.8325
0.9500	1.8788
0.9750	1.9263
1.0000	1.9750

41 lines.

The Approximated graph of the integral curve $y = \varphi(t) = t^2 + 1$: with $h = .025$.



Example 2

Consider the IVP

$$\begin{cases} \frac{dy}{dt} = -\cos t \\ y(0) = 1 \end{cases}$$

- Compute the analytic solution $y = \varphi(t)$ of the ODE, and evaluate $\varphi(\pi)$.
- Use Euler's Method to approximate the solution at $t = \pi$ with 30 steps. That means $h = \frac{\pi}{30} \approx .1047$
- Compare that actual value $\varphi(\pi)$ and the approximated value.

Solution:

The ODE can be solved by a simple antiderivative:

$$\begin{cases} y = \varphi(t) = -\int \cos t dt + c \\ y(0) = 1 \end{cases} \implies y = \varphi(t) = -\sin t + 1$$

So, $\varphi(\pi) = 1$.
Next, use Euler Method Approximation. We give two options:

- Use simple excel program.
- Use the Matlab program Euler14 that I will give you.

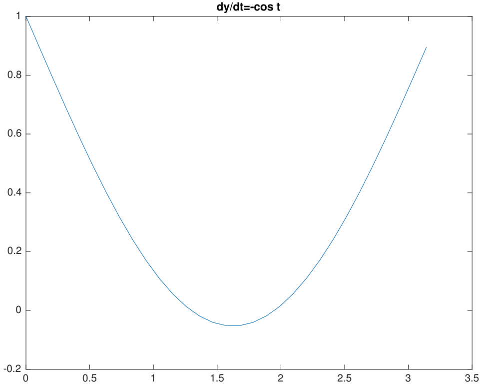
Euler14 Outputs

With $h = \frac{\pi}{30}$

t_i	y_i
0	1.0000
0.1047	0.8953
0.2094	0.7911
$\cdots$	$\cdots$
1.5708	-0.0514
1.6755	-0.0514
$\cdots$	$\cdots$
2.8274	0.5891
2.9322	0.6887
3.0369	0.7911
3.1416	0.8953

31 lines.

The Approximated graph of the integral curve $y = \varphi(t) = -\sin t + 1$: with $h = \frac{\pi}{30}$.



The table and the graph show negative values for $y = \varphi(t) = -\sin t + 1$, which shows the limitations of Euler

Example 3

Consider the following wo IVPs

$$\begin{cases} \frac{dy}{dt} = y - t \\ y(0) = 1 \end{cases} \quad nd \quad \begin{cases} \frac{dy}{dt} = y - t \\ y(0) = 0 \end{cases}$$

- Compute the analytic solution $y = \varphi(t)$ of the ODE, and evaluate $\varphi(1)$.
- Use Euler's Method to approximate the solutions at $t = 1$ with $h = .025$
- Compare that actual value $\varphi(1)$ and the approximated value.

Solution:

- The ODE can be written as: $\frac{dy}{dt} - y = -t$, which is linear.
- With integrating factor $\mu(t) = r^{-t}$, we have

$$e^{-t}y = \int -te^{-t}dt + c = te^{-t} + e^{-t} + c \Longrightarrow y = 1 + t + ce^{t}$$

- So, solutions, in these two cases:

$$\begin{cases} \text{If } y(0) = 1 & y = \varphi(t) = 1 + t & \varphi(1) = 2. \\ \text{If } y(0) = 0 & y = \psi(t) = 1 + t - e^{t} & \varphi(1) = 2 - e \end{cases}$$

Euler14 Outputs: The case $y(0) = 1$

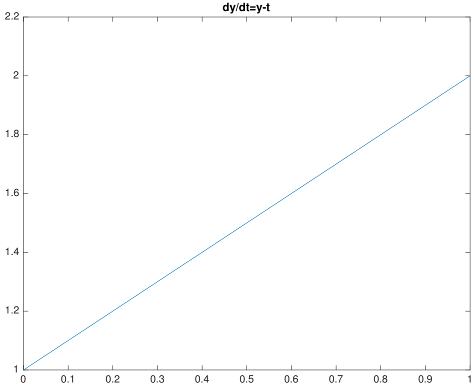
With $h = .025$, $y(0) = 1$

t_i	y_i
0	1.0000
0.0250	1.0250
0.0500	1.0500
0.0750	1.0750
$\cdots$	$\cdots$
0.9000	1.9000
0.9250	1.9250
0.9500	1.9500
0.9750	1.9750
1.0000	2.0000

41 lines.

The Case $y(0) = 1$

The Approximated graph of the integral curve $y = \varphi(t) = t + 1$:



Continued

This one is a straight line and matched perfectly, with actual values of $y = t + 1$.

Euler14 Outputs: The case $y(0) = 0$

With $h = .025, \ y(0) = 0$

t_i	y_i
0	0
0.0250	0
0.0500	−0.0006
0.0750	−0.0019
0.1000	−0.0038
...	...
0.9250	−0.5683
0.9500	−0.6057
0.9750	−0.6446
1.0000	−0.6851

41 lines.

Continued

Note $\psi(1) = 2 - e \approx -.7183.$

The Case $y(0) = 0$

The Approximated graph of the integral curve $y = \psi(t) = t + 1 - e^t$: