Chapter 2: First Order ODE
§2.4 Examples of such ODE Models

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28 January 2018

First Order ODE

Read Only Section!

- We recall the general form of the First Order DEs (FODE):

$$\frac{dy}{dt} = f(t, y) \qquad (1)$$

  where $f(t, y)$ is a function of both the independent variable t and the (unknown) dependent variable y.

- Purpose of this section is to give some examples of First Order (Linear) ODE Models.

Continued

- Also recall the form of the FOLE

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2}$$

- A general solution of (2) is

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + c\right] \tag{3}$$

where $\mu(t) = \exp\left(\int p(t)dt\right)$.

Recapitulation of Modeling from §1.1

- Recall (See §1.1), a Mathematical Model (in the context of DE), is a DE that describes (closely enough) some physical process.
- For example, we did modeling based on Newton's laws of motion. We got the equation $F = m\frac{dv}{dt}$.
- Then, we looked at the forces acting on the body (gravitational pull and drag) and refined the model to $m\frac{dv}{dt} = mg - \gamma v$.

Recapitulation of Modeling from §1.1

- Another example of modeling, we discussed was population growth.
  The basis of such modeling was the assumption that the rate at which the population size $p(t)$ changes is proportional to the current size $p(t)$. So, we modeled $\frac{dp}{dt} = rp$

A few points about modeling

- These examples demonstrate, that the DE that models such a phenomenon, is, actually, based on some theory or hypothesis.
- A mathematical model (the DE) is only an approximation to the actual physical system.
- A very satisfactory model (a DE), may not necessarily be easy to solve.
- So, we may opt for simpler models (that we can solve), at the cost of accuracy.
- First Order (Linear) ODEs are often simplest among all such Models. We consider some of them, in this section.

Compound Interest: Example 1

The model of continuous compound interest (and amortization) is a standard example of such a model.

Statement of the Model:

- An amount of money S_0 is invested in an interest paying account. Let $S(t)$ denote the account balance t years after the initial investment.
- The annual interest rate is r (in fraction not percent), compounded continuously.
- So, the rate of change in balance: $\frac{dS}{dt} = rS$.
- This is solved easily $S(t) = S_0 e^{rt}$.

Compare: Compounding m times a year

- Recall, if the interest is compounded m times, in a year, the balance $s(m,t) = S_0 \left(1+\frac{r}{m}\right)^{mt}$
- Intuitively, if m keeps increasing (monthly, daily, hourly, every minute etc) then, we should have $s(m,t) \approx S(t)$.
- In deed,

$$\lim_{m\to\infty} s(m,t) = \lim_{m\to\infty} S_0 \left(1+\frac{r}{m}\right)^{mt} = S(t).$$

Deposit or withdraw continuously

We change the problem:

- money is deposited/withdrawn at a constant rate k (k is negative, in case of withdrawal).
- So, rate of change $\frac{dS}{dt} = rS + k$, which is in the linear form $\frac{dS}{dt} - rS = k$. By the general solution (3), we have

$$S(t) = S_0 e^{rt} + \frac{k(e^{rt} - 1)}{r}$$

The first part is due to initial investment, second part is due to subsequence deposit/withdrawal.

Retirement Account

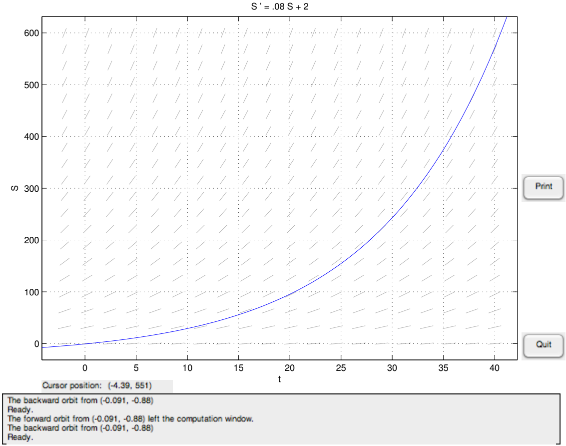
- Suppose someone opens a retirement account (IRA), at age 25. He makes an annual investments of $2,000 in a "continuous manner". (So, $k = 2000$) (Note unit of time t is "years"). Rate of interest is 8 percent. So, $r = .08$.
- Since, he makes no initial investment, $S_0 = 0$. Therefore,

$$S(t) = \frac{2000(e^{.08t} - 1)}{.08} = 25000(e^{.08t} - 1)$$

- **Question:** What would be the balance at his ager 70? **Solution:** At his age 70, $t = 70 - 25 = 45$. So, the balance would be

$$S(45) = 25000(e^{.08*45} - 1) = \$889,955.86$$

The Direction Fields and the integral curve



Example 2: Escape Velocity

Computing Escape velocity would be another standard example of such models.

Statement of the problem: A body of mass m is projected in the direction perpendicular to earth's surface, with initial velocity v_0. We write down a model for the velocity $v(t)$, at time t. We also compute the escape velocity.

Solution:

- As is standard, g denotes the acceleration due to gravity, at the surface of earth. Let R denote the radius of earth.
- The vertical line through the point of projection will denote the x-axis and positive direction is away from the center of earth. At the point of projection, $x = 0$.

The Model: Escape Velocity

- It is known from physics, the gravitational force acting on a body is inversely proportional to the distance from the center of earth. So, when the body is at the position x (i. e. at height x), gravitational pull is given by $w(x) = -\frac{k}{(x+R)^2}$, where k is a constant.
- Also, at the surface of earth $w(0) = -mg$.
- Therefore $k = mgR^2$ and $w(x) = -\frac{mgR^2}{(x+R)^2}$
- Since no other force is acting on the body (no drag) the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(x+R)^2} \quad Or \quad \frac{dv}{dt} = -\frac{gR^2}{(x+R)^2} \qquad (4)$$

The Solution: Escape Velocity

- We have three variables, which is not convenient. But $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$. So, equation 4 reduces to

$$v\frac{dv}{dx} = -\frac{gR^2}{(x+R)^2} \tag{5}$$

- We use separation of variables: $\int v dv = -\int \frac{gR^2}{(x+R)^2} dx + c$
- So, $\frac{v^2}{2} = \frac{gR^2}{(x+R)} + c$
- Write $v(0) = v_0$. Then, $c = \frac{v_0^2}{2} - gR$. So, the solution is

$$\frac{v^2}{2} = \frac{gR^2}{(x+R)} + \left(\frac{v_0^2}{2} - gR\right)$$

The Solution: Escape Velocity

- So,

$$v = \pm\sqrt{\frac{2gR^2}{(x+R)} + (v_0^2 - 2gR)}$$

- Since $v(0) = +v_0$, we actually have

$$v = \sqrt{\frac{2gR^2}{(x+R)} + (v_0^2 - 2gR)} \qquad (6)$$

as a function of x.

Required initial velocity: Escape Velocity

Find the initial velocity required to reach altitude ζ.

- If maximum height is ζ, then $v(\zeta) = 0$. Therefore

$$0 = \sqrt{\frac{2gR^2}{(\zeta + R)} + (v_0^2 - 2gR)} \Longrightarrow v_0 = \sqrt{2gR - \frac{2gR^2}{(\zeta + R)}}$$

OR

$$v_0 = \sqrt{\frac{2gR\zeta}{(\zeta + R)}}$$

Required initial velocity to "Escape"

Find the initial velocity to "escape":

- Let v_e denote the escape velocity. In deed,

$$v_e = \lim_{\zeta \to \infty} v_0 = \lim_{\zeta \to \infty} \sqrt{\frac{2gR\zeta}{(\zeta + R)}} = \sqrt{2gR}$$

Example 3: Model for Chemicals in a Pond

There are some examples in the literature (Textbooks) about concentration of certain chemicals in a solution. This goes with estimating impurities in water and purifications. The textbook of Boyce and Diprima (§2.3) has a good collection of such examples.

- **Statement:** A pond has 10 million (i.e. 10^7) gallons of water. Five million (i.e. $5 * 10^6$) gallons of water flows in to the pond, per year, and water flows out at the same rate.
  The incoming water contains some chemicals, with $\gamma(t) = 2 + \sin 2t \; g/gal$. In the next frame, we model this flow process.

Example 3: Chemicals in a Pond

- Let $Q(t)$ be the quantity of the chemicals in the pond.
- The rate of change $\frac{dQ}{dt} = \text{rate in} - \text{rate out}$
- The $\text{rate in} = 5 * 10^6 \gamma(t) = 5 * 10^6 (2 + \sin 2t)$.
- The $\text{rate out} =$

$$(5 * 10^6) * concentration = (5 * 10^6) \frac{Q(t)}{10^7} = .5Q(t)$$

- So, the model

$$\frac{dQ}{dt} = 5 * 10^6 (2 + \sin 2t) - .5Q(t)$$

Solution: Example 3

- We rewrite the DE in the linear form

$$\frac{dQ}{dt} + .5Q(t) = 5 * 10^6(2 + \sin 2t)$$

- The integrating factor $\mu(t) = \exp(\int .5dt) = e^{.5t}$.
- By (3), the general solution is

$$Q(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + c\right]$$

$$= e^{-.5t}\left[\int e^{.5t}5 * 10^6(2 + \sin 2t)dt + c\right]$$

Continued

- $Q(t) = e^{-.5t}\left[5 * 10^6 \int e^{.5t}(2 + \sin 2t)dt + c\right]$

$$= e^{-.5t}\left[5 * 10^6 \left(4e^{.5t} + \int e^{.5t} \sin 2t dt\right) + c\right]$$

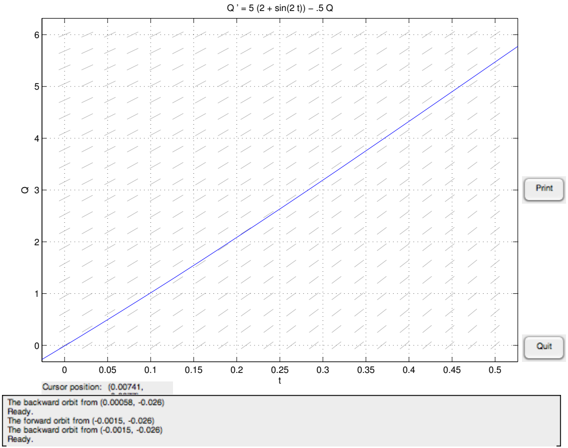
$$= e^{-.5t}\left[5 * 10^6 \left(4e^{.5t} + \frac{1}{17}\left[2\sin 2t \ e^{.5t} - 8\cos 2t \ e^{.5t}\right]\right) + c\right]$$

You expected to be able to compute the second integral (see below).

Continued

- $Q(t) = 10^6 \left[\left(20 + \frac{10}{17} \sin 2t - \frac{40}{17} \cos 2t \right) + ce^{-.5t} \right]$
- Since $Q(0) = 0$, we have $c = \frac{40}{17} - 20 = -\frac{300}{17}$.
- So, $Q(t) = 10^6 \left[\left(20 + \frac{10}{17} \sin 2t - \frac{40}{17} \cos 2t \right) - \frac{300}{17} e^{-.5t} \right]$

The Direction Fields and the integral curve. (Unit in million gallons)



Compute $I = \int e^{.5t} \sin 2t dt$

- Write $I = \int e^{.5t} \sin 2t dt$. Then, $I = 2 \int \sin 2t \ de^{.5t} dt$

$$= 2 \left[\sin 2t \ e^{.5t} - 2 \int e^{.5t} \cos 2t \ dt \right]$$

$$= 2 \left[\sin 2t \ e^{.5t} - 4 \int \cos 2t \ de^{.5t} \right]$$

$$= 2 \left[\sin 2t \ e^{.5t} - 4 \left(\cos 2t \ e^{.5t} + 2I \right) \right]$$

$$I = \frac{1}{17} \left[2 \sin 2t \ e^{.5t} - 8 \cos 2t \ e^{.5t} \right]$$

Example 4

Here is a similar example. A container mixes salt and water.

- $Q(t)$ =quantity of salt (in lbs) in 100 gallon water in a container.
- $Q(0) = Q_0$ is the initial quantity of salt.
- Water with of concentration .25 lb/gal is entering the container at the rate of r gal/min. And, well-stirred mixture is draining out at the same rate.

Example 4

First, we set up the initial value problem:

- The rate of change of quantity of salt is $\frac{dQ}{dt}$.
- Now $\frac{dQ}{dt} = \text{rate in} - \text{rate out}$
- $\text{rate in} = .25r$
- r gallons of mixture is draining out, which has $\frac{Q(t)}{100}r$ lbs of salt. So, $\text{rate out} = \frac{Q(t)}{100}r$
- Therefore $\frac{dQ}{dt} = \text{rate in} - \text{rate out} = .25r - \frac{Q(t)}{100}r$
- The initial value problem is

$$\begin{cases} \frac{dQ}{dt} & = .25r - \frac{Q(t)}{100}r \\ Q(0) & = Q_0 \end{cases}$$

Continued: Intuition

- Intuitively, it seems that, in the limit, concentration of salt will be the same as that of incoming mixture, i.e .25 lb/gal.
- Really?

Continued: Solution

- The DE can be rewritten in the linear form: $\frac{dQ}{dt} + \frac{rQ(t)}{100} = .25r$
- Then integrating factor $\mu(t) = \exp(\int \frac{r}{100} dt) = e^{\frac{rt}{100}}$
- By equation 3, a general solution is

$$Q(t) = \frac{1}{\mu(t)} \left[\int \mu(t) g(t) dt + c \right]$$

$$= e^{-\frac{rt}{100}} \left[\int e^{\frac{rt}{100}} (.25r) dt + c \right] = e^{-\frac{rt}{100}} \left[\frac{e^{\frac{rt}{100}}}{\frac{r}{100}} (.25r) + c \right]$$

Continued

- $Q(t) = 25 + ce^{-\frac{rt}{100}}$
- By the initial condition $Q(0) = Q_0$, we have $c = Q_0 - 25$
- So, the solution of the initial value problem is:

$$Q(t) = 25 + (Q_0 - 25)e^{-\frac{rt}{100}}$$

Continued

- The limiting amount:

$$Q_l = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left(25 + (Q_0 - 25)e^{-\frac{rt}{100}}\right) = 25$$

This is consistent with our intuition.

Continued

- Suppose $r = 3$ and $Q_0 = 2Q_l$. Then $Q_0 = 50$.
- Then, $Q(t) = 25 + (Q_0 - 25)e^{-\frac{rt}{100}} = 25 + 25e^{-\frac{3t}{100}}$
- We find the T, when $Q(t)$ with within 2 percent of Q_l. That means, $Q(T) = 1.02 * 25 = 25.5$. So,

$$25.5 = Q(T) = 25 + 25e^{-\frac{3T}{100}} \Longrightarrow \ln .02 = -\frac{3T}{100} \Longrightarrow$$

$$T = 130.4 \ min$$

The Direction Fields and the integral curve