Chapter 6: The Laplace Transform
§6.1 Definition of Laplace Transform

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The Laplace Transform: Definition

The Laplace Transform is another tool to solve differential equations, which we define next.

Definition Suppose $f(t)$ is a function on $t \geq 0$. The Laplace Transform of f is defined to be the function

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

For a given $f(t)$, $F(s)$ exists for s in certain interval. This is derived from theorems on existence of definite integrals. A sufficient conditions are given below.

Existence theorem

Theorem 6.1.1 Suppose f is a piecewise continuous on an interval $0 \leq t \leq \alpha$, for some $\alpha > 0$. Suppose, there are positive constants κ, β, λ with $\kappa > 0$, $\beta > 0$ such that

$$|f(t)| \leq \kappa e^{\lambda t} \qquad \forall\ t \geq \beta$$

Then, the Laplace transform $F(s) = \mathcal{L}\{f(t)\}(s)$ exists on the interval $\lambda < s$.

Important Integration Techniques

- In this chapter, Integration by parts is used extensively:

$$\int f(t)d(g(t)) = f(t)g(t) - \int g((t)f'(t)dt$$

- In particular, the formulas

$$\begin{cases} \int e^{\lambda t} \cos \mu t dt = e^{\lambda t} \frac{\mu \sin \mu t + \lambda \cos \mu t}{\lambda^2 + \mu^2} \\ \\ \int e^{\lambda t} \sin \mu t dt = e^{\lambda t} \frac{\lambda \sin \mu t - \mu \cos \mu t}{\lambda^2 + \mu^2} \end{cases} \qquad (2)$$

Formula 1

Derive the Formula

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad s > a.$$

Proof. With $f(t) = e^{at}$, for $s > a$, we have

$$\mathcal{L}\{f(t)\}(s) := F(s) = \int_0^\infty e^{at}e^{-st}dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=0}^{\infty} = \frac{1}{s-a}$$

Formula 2

For the constant function $f(t) = 1$, derive the Formula

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad s > 0.$$

Proof. Follows from Example 1, with $a = 0$.
One can redo it, directly. ■

Formula 3

For $\mu \neq 0$, derive

$$\begin{cases} \mathcal{L}\{\cos \mu t\}(s) = \frac{s}{s^2+\mu^2} & s > 0. \\ \mathcal{L}\{\sin \mu t\}(s) = \frac{a}{s^2+\mu^2} & s > 0. \end{cases}$$

Proof. We derive the first one only. By definition,

$$\mathcal{L}\{\cos \mu t\}(s) = \int_0^\infty \cos \mu t e^{-st} dt$$

Now, if follows from (2). ∎

Formula 4

Let $f(t) = \begin{cases} 1 & if \ 0 \leq t \leq 1 \\ 0 & otherwise \end{cases}$.

$$\text{Derive} \quad \mathcal{L}\{f(t)\} = \frac{1 - e^{-s}}{s} \qquad s > 0.$$

Proof.

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-st} dt + \int_1^\infty 0 dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_{t=0}^1 = \frac{e^{-s}}{-s} + \frac{1}{s} = \frac{1 - e^{-s}}{s}$$

Formula 5

Derive $\mathcal{L}(\{t^2\}(s) = \frac{2}{s^3} \quad s > 0.$

Proof. By definition, the Laplace Transform

$$F(s) = \mathcal{L}\{t^2\} = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} t^2 dt$$

$$= \lim_{A \to \infty} \int_0^A e^{-st} t^2 dt = \lim_{A \to \infty} \left[\frac{1}{-s} \int_0^A t^2 de^{-st} \right]$$

$$= -\frac{1}{s} \lim_{A \to \infty} \left[\left[t^2 e^{-st} \right]_{t=0}^A - 2 \int_0^A t e^{-st} dt \right]$$

- So,

$$F(s) = -\frac{1}{s}\lim_{A\to\infty}\left[A^2e^{-sA} - 0\right] + \frac{2}{s}\lim_{A\to\infty}\left[\int_0^A te^{-st}dt\right]$$

- If $s < 0$, then the first limit is $\pm\infty$. So, now on we assume $s > 0$.
- When $s > 0$, the first limit:

$$\lim_{A\to\infty} A^2e^{-sA} = \lim_{A\to\infty}\frac{A^2}{e^{sA}} = 0$$

Why?: You need to know this. Two ways to see it:

  - Informally or Intuitively: Exponential growth is faster than polynomial growth.
  - Formally: Use L'Hôspital rule

- So,

$$F(s) = \frac{2}{s} \lim_{A\to\infty} \left[\int_0^A te^{-st} dt \right] = \frac{-2}{s^2} \lim_{A\to\infty} \left[\int_0^A tde^{-st} \right]$$

$$= \frac{-2}{s^2} \lim_{A\to\infty} \left[\left[te^{-st} \right]_{t=0}^A - \int_0^A e^{-st} dt \right]$$

$$= \frac{-2}{s^2} \lim_{A\to\infty} \left[Ae^{-sA} - 0 \right] + \frac{2}{s^2} \lim_{A\to\infty} \left[\int_0^A e^{-st} dt \right]$$

The first limit is zero, by same reasoning as above.

- So,

$$F(s) = \frac{2}{s^2} \lim_{A\to\infty}\left[\int_0^A e^{-st}dt\right] = \frac{2}{s^2} \lim_{A\to\infty}\left[\left[\frac{e^{-st}}{-s}\right]_{t=0}^{A}\right]$$

$$= \frac{2}{s^2} \lim_{A\to\infty}\left[\frac{e^{-sA}}{-s} + \frac{1}{s}\right] = \frac{2}{s^3} \qquad s > 0.$$

Alternative Informal Method: By Substituting ∞

- Try to substitute $A = \infty$, if you can get away with it:

$$F(s) = \mathcal{L}\{t^2\} = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} t^2 dt$$

$$= \int_0^\infty e^{-st} t^2 dt = \frac{1}{-s} \int_0^\infty t^2 de^{-st}$$

$$= -\frac{1}{s} \left[\left[t^2 e^{-st} \right]_{t=0}^{\infty} - 2 \int_0^\infty te^{-st} dt \right] \qquad \text{Assume } s > 0$$

$$= -\frac{1}{s} \left[\left[\infty^2 e^{-\infty} - 0 \right] - 2 \int_0^\infty te^{-st} dt \right]$$

- $$= [0 - 0] + \frac{2}{s}\int_0^\infty te^{-st}dt = \frac{2}{s}\int_0^\infty te^{-st}dt$$
$$= \frac{-2}{s^2}\left[\int_{t=0}^\infty tde^{-st}\right] = \frac{-2}{s^2}\left[\left[te^{-st}\right]_{t=0}^\infty - \int_0^\infty e^{-st}dt\right]$$
$$= \frac{-2}{s^2}\left[\left[\infty e^{-\infty} - 0\right] + \int_0^\infty e^{-st}dt\right]$$
$$= \frac{-2}{s^2}\left[[0 - 0] - \int_0^\infty e^{-st}dt\right] = \frac{2}{s^2}\int_0^\infty e^{-st}dt$$

- So,

$$F(s) = \frac{2}{s^2}\left[\left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty}\right]$$

$$= \frac{2}{s^2}\left[\frac{e^{-\infty}}{-s} + \frac{1}{s}\right] = \frac{2}{s^2}\left[0 + \frac{1}{s}\right] = \frac{2}{s^3} \qquad s > 0.$$

Formula 6

For any integer $n \geq 1$, derive

$$\mathcal{L}(\{t^n\}(s) = \frac{n!}{s^{n+1}} \quad s > 0$$

Proof. Use Integration by parts, similar to the the case $f(t) = t^2$, and induction. ■

Example 1

Compute the Laplace transform of $f(t) = te^{\alpha t}$.

- By definition

$$\mathcal{L}\{te^{\alpha t}\} = \int_0^\infty e^{-st} te^{\alpha t} dt = \int_0^\infty te^{-(s-\alpha)t} dt$$

- Use integration by parts:

$$\mathcal{L}\{te^{\alpha t}\} = \frac{1}{-(s-\alpha)} \int_0^\infty t de^{-(s-\alpha)t}$$

$$= \frac{-1}{(s-\alpha)} \left[\left[te^{-(s-\alpha)t} \right]_{t=0}^\infty - \int_0^\infty e^{-(s-\alpha)t} dt \right]$$

- Now assume $s > \alpha$. So,

$$\mathcal{L}\{te^{\alpha t}\} = \frac{-1}{(s-\alpha)}\left[\left[\infty e^{-\infty} - 0\right] - \int_0^\infty e^{-(s-\alpha)t}dt\right]$$

$$\frac{-1}{(s-\alpha)}\left[\left[0 - 0\right] - \int_0^\infty e^{-(s-\alpha)t}dt\right]$$

$$= \frac{1}{(s-\alpha)}\int_0^\infty e^{-(s-\alpha)t}dt = \frac{-1}{(s-\alpha)^2}\left[e^{-(s-\alpha)t}\right]_{t=0}^{\infty}$$

$$\frac{-1}{(s-\alpha)^2}\left[e^{-\infty} - 1\right] = \frac{1}{(s-\alpha)^2}$$

Example 2

Compute the Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

- By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^1 te^{-st}dt + \int_1^\infty 0dt$$

$$= \int_0^1 te^{-st}dt = \frac{1}{-s}\int_{t=0}^1 tde^{-st}dt$$

$$= \frac{1}{-s}\left[\left[te^{-st}\right]_{t=0}^1 - \int_{t=0}^1 e^{-st}dt\right]$$

- $$= \frac{1}{-s}\left[\left[e^{-s} - 0\right] + \frac{1}{s}\left[e^{-st}\right]_{t=0}^{1}\right]$$
$$= \frac{1}{-s}\left[e^{-s} + \frac{1}{s}\left[e^{-s} - 1\right]\right]$$
$$= \frac{1 - e^{-s}(s+1)}{s^2}$$

Linearity Property

- Laplace transform is a Linear Operator. That means

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} \qquad (3)$$

  for any two functions f_1, f_2 and constants c_1, c_2. The proof of (3) follows immediately from the linearity property of integration.
- Often, the Laplace Transform $\mathcal{L}\{f(t)\}$ is computed, by using a combination of linearity and the Standard Formulas. We gave a few such Formulas above. Charts of standard formulas is available in the internet.
- The following is an application the linearity property.

Example 2

Find the Laplace transform of

$$f(t) = \begin{cases} 3\sin 2t - 7t & \textit{if } 0 \leq t \leq 1 \\ 3\sin 2t & \textit{otherwise} \end{cases}$$

- We have

$$f(t) = 3\sin 2t - 7g(t) \text{ where } g(t) = \begin{cases} t & \textit{if } 0 \leq t \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$

- By linearity property of Laplace transform and using the Formulas and Examples above, we have

$$\mathcal{L}\{f(t)\} = 3\mathcal{L}\{\sin 2t\} - 7\mathcal{L}\{g(t)\}$$

$$= 3\frac{2}{s^2+4} - 7\frac{1-e^{-s}(s+1)}{s^2} \qquad s > 0.$$