Chapter 6: The Laplace Transform §6.3 Step Functions and Dirac δ

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Step Function

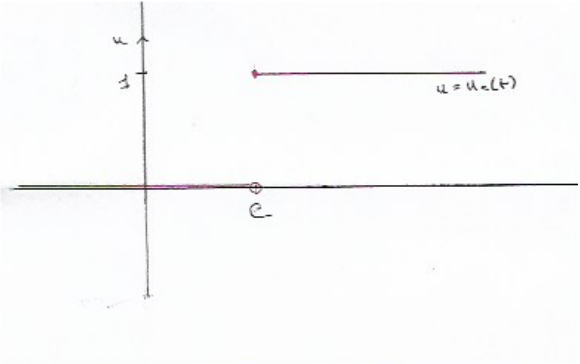
- Definition: Suppose c is a fixed real number. The unit step function u_c is defined as follows:

$$u_c(t) = \begin{cases} 0 & if \;\; t < c \\ 1 & if \;\; c \leq t \end{cases} \tag{1}$$

- If follows immediately,

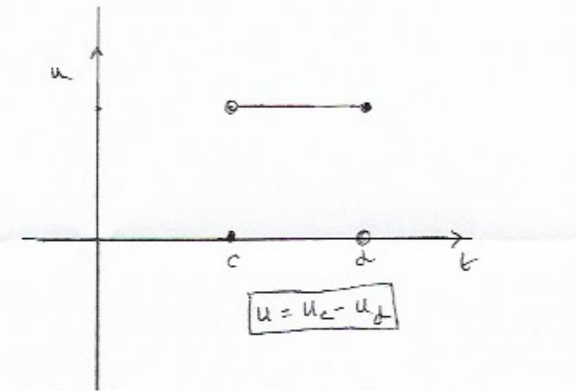
$$1 - u_c(t) = \begin{cases} 1 & if \;\; t < c \\ 0 & if \;\; c \leq t \end{cases}$$

The Graph of $u = u_c(t)$



- It also follows: For two real numbers $c < d$, we have

$$u_c - u_d(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } c \leq t < d \\ 0 & \text{if } d \leq t \end{cases} \qquad (2)$$



Impulse Function Dirac $\delta(t)$

- In physics, to represent a unit impulse, the Dirac Delta Function is defined, by the following two conditions:

$$\begin{cases} \delta(t) = 0 \quad \textit{if} \ \ t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)dt = 1 \end{cases} \tag{3}$$

This may appear to be an unusual way to define a function, because $\delta(0)$ is not given in any form. For of this reason, in Algebra, this will not qualify as a function, .

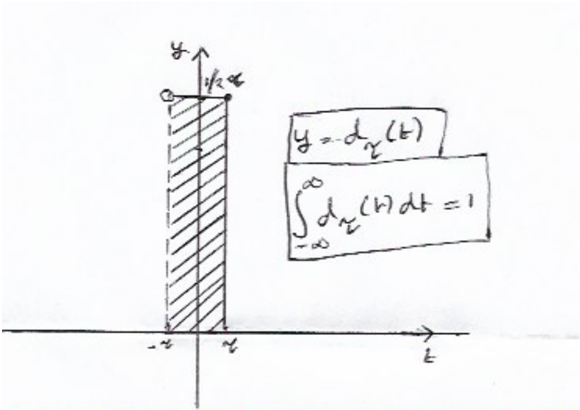
Approximation to Dirac $\delta(t)$

For $\tau > 0$, define a function d_τ, as follows:

$$d_\tau(t) = \frac{1}{2\tau}(u_{-\tau} - u_\tau)(t) = \begin{cases} 0 & \text{if } t < -\tau \\ \frac{1}{2\tau} & \text{if } -\tau \leq t < \tau \\ 0 & \text{if } \tau \leq t \end{cases}$$

Refer to the graph in the next frame.

Continued



Graph of $y = d_\tau(t)$:

Continued

We have the following:

- $\int_{-\infty}^{\infty} d_\tau(t)dt = 1$ for all $\tau \neq 0$.

$$\text{And,} \quad \lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t)dt = 1$$

- $\lim_{\tau \to 0} d_\tau(t) = \begin{cases} 0 & if \;\; t \neq 0 \\ \infty & if \;\; t = 0 \end{cases}$

So, one can view

$$\delta = \lim_{\tau \to 0} d_\tau$$

Laplace Transform of u_c

- For $c \geq 0$, the Laplace Transform

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \qquad s > 0.$$

- Proof. By definition $\mathcal{L}\{u_c(t)\} =$

$$\int_0^\infty e^{-st} u_c(t)dt = \int_0^c e^{-st} u_c(t)dt + \int_c^\infty e^{-st} u_c(t)dt$$

$$= \int_0^c 0dt + \int_c^\infty e^{-st} dt = \frac{e^{-cs}}{s}$$

Translation of a Function

- Given a function $f(t)$ on the domain $t \geq 0$, define

$$g(t) = \begin{cases} 0 & if \;\; t < c \\ f(t-c) & if \;\; t \geq c \end{cases}$$

- Write $g(t) = u_c(t)f(t-c)$.
- $g(t)$ is translation of f to the right by c.

Laplace Transform and Translation

Theorem 6.3.1 Suppose $c > 0$ and $F(s) = \mathcal{L}\{f(t)\}$.

$$\text{Then,} \quad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$$\text{Therefore,} \quad \mathcal{L}^{-1}\{e^{-cs}\mathcal{L}\{f(t)\}\} = u_c(t)f(t-c).$$

Proof. $\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty u_c(t)f(t-c)e^{-st}dt$

$$= \int_c^\infty f(t-c)e^{-st}dt = \int_c^\infty f(x)e^{-s(x+c)}dx = e^{-cs}F(s)$$

■

Laplace Transform and Translation

Theorem 6.3.2 Suppose $c > 0$ and $F(s) = \mathcal{L}\{f(t)\}$. Then,

$$\mathcal{L}\{e^{ct} f(t - c)\} = F(s - c)$$

Therefore,

$$\mathcal{L}^{-1}\{F(s - c)\} = e^{ct} f(t - c).$$

Proof. Similar to the above. ■

Remark. The formulas in these two theorems are in the charts.

Example 1

Express the following function in terms of step functions:

$$f(t) = \begin{cases} 2 & \textit{if } 0 \leq t < 1 \\ -2 & \textit{if } 1 \leq t < 2 \\ 2 & \textit{if } 2 \leq t < 3 \\ -2 & \textit{if } 3 \leq t < 4 \\ 0 & \textit{if } t \geq 4 \end{cases}$$

- Use the (2) for $u_c - u_d$.
- The first line is given by $(u_0 - u_1)$, the second line is given by $-(u_1 - u_2)$, the third line is given by $(u_2 - u_3)$, the fourth line is given by $-(u_3 - u_4)$.

- So,

$$f(t) = 2(u_0 - u_1) - 2(u_1 - u_2) + 2(u_2 - u_3) - 2(u_3 - u_4)$$

$$= 2u_0 - 4u_1 + 4u_2 - 4u_3 + 2u_4$$

Example 2

Compute the Laplace transform of the function:

$$f(t) = \begin{cases} 0 & \textit{if} \;\; t < 2 \\ t^2 - 4t + 5 & \textit{if} \;\; t \geq 2 \end{cases}$$

To use theorem 6.3.1 (or the Charts), rewrite $f(t)$:

$$f(t) = \begin{cases} 0 & \textit{if} \;\; t < 2 \\ (t-2)^2 + 1 & \textit{if} \;\; t \geq 2 \end{cases}$$

- With $g(t) = t^2 + 1$, we have

$$f(t) = u_2(t)g(t-2)$$

- By Theorem 6.3.1 (Use Charts),

$$\mathcal{L}\{u_2(t)g(t-2)\} = e^{-2s}\mathcal{L}\{t^2+1\}$$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$$

Example 3

Compute the inverse Laplace transform of the function:

$$F(s) = \frac{(s-2)e^{-2s}}{s^2 - 4s + 5}$$

- Write $F(s) = e^{-2s}H(s)$ where $H(s) = \frac{(s-2)}{s^2-4s+5}$.
- Also write $\mathcal{L}^{-1}\{H(s)\} = h(t)$
- By Theorem 6.3.1 (Use Charts)

$$\mathcal{L}^{-1}\{F(s)\} = u_2(t)h(t-2) \tag{4}$$

- $$h(t)=\mathcal{L}^{-1}\{H(s)\}=\mathcal{L}^{-1}\left\{\frac{(s-2)}{s^2-4s+5}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+1}\right\}=e^{2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \qquad \text{(Use Chart)}$$
$$=e^{2t}\cos t$$

- By (4)

$$\mathcal{L}^{-1}\{F(s)\} = u_2(t)h(t-2)\} = u_2(t)e^{2(t-2)}\cos(t-2)$$

The Dirac Delta $\delta(t)$, represents Unit Impulse, at time $t=0$. So, the Unit Impulse at time $t=t_0$, is represented, by the translation $\delta(t-t_0)$, which we denote by $\delta^{t_0}(t) := \delta(t-t_0)$. More directly, define

$$\begin{cases} \delta^{t_0}(t) = 0 & \textit{if } \ t \neq t_0 \\ \int_{-\infty}^{\infty} \delta^{t_0}(t)dt = 1 & \end{cases}$$

Laplace Transform of $d_\tau(t - t_0)$

Fix, $t_0 > 0$ and $\tau > 0$ such that $t_0 > \tau$. Define,

$$d_\tau^{t_0}(t) := d_\tau(t - t_0) = \begin{cases} 0 & \text{if } t < t_0 - \tau \\ \frac{1}{2\tau} & \text{if } t_0 - \tau \leq t < t_0 + \tau \\ 0 & \text{if } t_0 + \tau \leq t \end{cases}$$

So, $d_\tau^{t_0} = \frac{1}{2\tau}\left(u_{t_0-\tau} - u_{t_0+\tau}\right)$. So, the Laplace Transform

$$\mathcal{L}\{d_\tau^{t_0}\}(s) = \frac{1}{2\tau}\left(\mathcal{L}\{u_{t_0-\tau}\}(s) - \mathcal{L}\{u_{t_0+\tau}(s)\}\right)$$

$$= \frac{1}{2\tau}\left(\frac{e^{-(t_0-\tau)s}}{s} - \frac{e^{-(t_0+\tau)s}}{s}\right) = \frac{e^{-t_0 s}}{2s}\left(\frac{e^{s\tau} - e^{-s\tau}}{\tau}\right)$$

Laplace Transform of Dirac $\delta^{t_0}(t)$

Theorem For time $t_0 > 0$, the Laplace Transform of Dirac $\delta^{t_0}(t)$ is:

$$\mathcal{L}\{\delta^{t_0}(t)\} = e^{-t_0 s}$$

Outline of the Proof. We can view

$$\lim_{\tau\to 0} d_\tau^{t_0} = \delta^{t_0}. \qquad \text{So,}$$

$$\mathcal{L}\{\delta^{t_0}\} = \lim_{\tau\to 0}\mathcal{L}\{d_\tau^{t_0}\} = \lim_{\tau\to 0}\left(\frac{e^{-t_0 s}}{2s}\left(\frac{e^{s\tau}-e^{-s\tau}}{\tau}\right)\right) = e^{-t_0 s}$$

To compute this limit, one can use L'Hospital's Rule.