Chapter I: Introduction
§1.2 Solving Some DE
§1.3: Classification of DEs

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Equations from §1.1

We recall the equations discussed in §1.1.

- Falling Object Models:

$$m\frac{dv}{dt} = mg - \gamma v \tag{1}$$

$$10\frac{dv}{dt} = 98 - 2v \quad or \quad \frac{dv}{dt} = 9.8 - .2v \tag{2}$$

Continued

- Population Growth Model:

$$\frac{dp}{dt} = rp \tag{3}$$

$$\frac{dp}{dt} = .5p - 450 \tag{4}$$

- General First Order Equations:

$$\frac{dy}{dt} = f(t, y) \quad \text{where } f \ \text{ is a function of } \ t, y. \tag{5}$$

The equations in §1.1 have been fairly simple, in the sense:

- All the DEs are of the form (5): $\frac{dy}{dt} = f(t, y)$. It involves only first derivative; and no higher order derivatives.
- For these DEs (1, 2, 3), the right side $f(t, y)$ are linear.
- Solving such DEs (5), mainly, involves nothing more than revisiting antiderivatives.

Solving the Growth Model

- We solve the population growth model (4)

$$\frac{dp}{dt} = .5p - 450 \quad \Longrightarrow \frac{dp}{.5p - 450} = dt \tag{6}$$

- $\int \frac{dp}{.5p-450} = \int dt + C$, where C is an arbitrary constant.
- Substituting $u = .5p - 450$ we get

$$\frac{du}{u} = .5 \int dt + C \quad Or \quad \ln|u| = .5t + C$$

$$|.5p - 450| = e^{.5t+C} = ce^{.5t} \quad Or \quad p = 900 + ce^{.5t}$$

wher $c := \pm e^C > 0$ is an arbitrary constant.

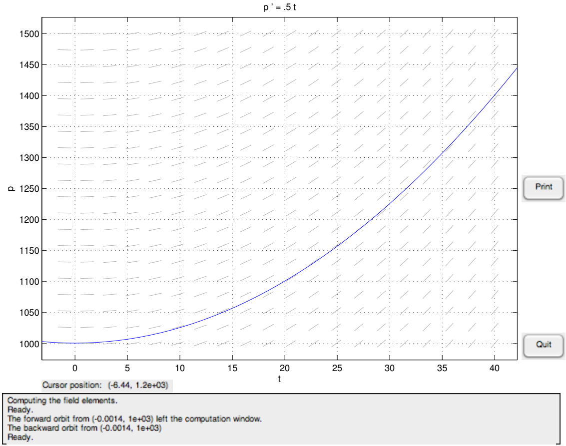
Initial Value

- $p = 900 + ce^{.5t}$ is a solution of (6), for all values of c. This would be called the General solution
- In the absence of additional information, we cannot determine the value of c.
- Such extra information is provided, often, by giving the population size $p(t_0)$ at a particular time t_0. For example, it may be given that $p(0) = 1000$. Such information, is called an initial value.

- In case, $p(0) = 1000$, we have

$$1000 = p(0) = 900 + c, \quad c = 100$$

  Finally, our particular solution is $p = 900 + 100e^{.5t}$
- In the next frame, compare the direction fields of the DE (4), with this solution $p = 900 + 100e^{.5t}$.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Cursor position: (-6.44, 1.2e+03)
Computing the field elements.
Ready.
The forward orbit from (-0.0014, 1e+03) left the computation window.
The backward orbit from (-0.0014, 1e+03)
Ready.

Solving such general equations

More generally, consider the initial value problem:

$$\begin{cases} \frac{dy}{dt} & = ay - b \\ y(0) & = y_0 \end{cases} \qquad a, b \quad \text{are constants, and} \qquad (7)$$

y_0 is (an) initial value of y, at time $t = 0$.

continued

(Trivial cases):

- If $a = 0$ then the equation is rewritten as

$$\begin{cases} \frac{dy}{dt} & = -b \\ y(0) & = y_0 \end{cases} \qquad Solution : exercise$$

- If $ay - b = 0$ then, $y = y(t) = b/a$ and there is nothing to solve. We have

$$\begin{cases} \frac{dy}{dt} & = 0 \\ y(0) & = y_0 \end{cases} \qquad Solution : (Answer : y = y_0 = b/a)$$

continued

(The Non-Trivial case):

$$\begin{cases} \frac{dy}{dt} & = ay - b \\ y(0) & = y_0 \end{cases} \qquad a \neq 0, ay - b \neq 0 \tag{8}$$

We proceed as in the growth model equation:

- We have $\frac{dy}{ay-b} = dt$. So, $\int \frac{dy}{ay-b} = \int dt + C$, where C is an arbitrary constant.
- So,

$$\int \frac{dy}{y - \frac{b}{a}} = a \int dt + C \Longrightarrow \ln \left| y - \frac{b}{a} \right| = at + C$$

continued

- Taking exponential: The general solution of (8) is:
  $y - \frac{b}{a} = ce^t \qquad where \quad c = \pm e^C$ is also arbitrary
- $c = 0$ corresponds to the equilibrium solution $y = \frac{b}{a}$.
- Using the initial value $y(0) = y_0$: $y_0 - \frac{b}{a} = c$
- So, the final solution of the initial value problem (8) is:
  $y - \frac{b}{a} = \left[y_0 - \frac{b}{a}\right] e^{at}$. Which is

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a}\right] e^{at} \tag{9}$$

Standard Examples

Following are some of the standard examples, available in the textbooks:

- Mass of decaying mass (usually radio).
  The Population Growth Model above, the growth or amortization of an interest paying account would be analogous.
- Motion of an ejected or falling body.
- Concentration of salt (or impurity) in a solution that is constantly diluted.

We discuss such examples subsequently.

Example 1: Decaying Mass

Statement: Let $Q(t)$ denote the mass of some radio-active substance, at time t. It is known that such substances disintegrates at a rate proportional to the current mass $Q(t)$. Write down a model, for this phenomenon.

- The rate of disintegration, at time t would be $\frac{dQ}{dt}$. According to the above stated model, $\frac{dQ}{dt}$ is proportional to $Q(t)$.
- So, the model is $\frac{dQ}{dt} = -rQ(t)$, for some constant $r > 0$.
- By (8) and solution 9, with $b = 0, a = -r$, we have

$$Q(t) = Q(0)e^{-rt}$$

Continued

Statement: Now suppose initial mass is 1000 grams, which reduces to 900 grams in 10 hours. Compute r.

- We are also given $Q(0) = 1000$ *gram* and $Q(10) = 900$ grams (Unit of time used is "hours").
- So, we have

$$900 = 1000e^{-10r}. \quad r = -\frac{\ln(.9)}{10} = .0105$$

- So, $Q(t) = 1000e^{-.0105t}$.

Example 2: Motion of a Falling Body

Statement: A missile has a vertical motion and horizontal motion. In this example, we only consider the vertical motion of such a missile. Suppose such a missile of mass 1000 kg, is projected and the vertical drag is proportional to square of the velocity. We formulate the model for vertical velocity.

- $v(t)$ will denote the vertical of the missile, at time t.
- The model of the falling body DE (1) was modified, by changing model on drag. By the stated model, the drag$= \gamma v^2$.
- So, the new model DE is

$$m\frac{dv}{dt} = mg - \gamma v^2 \tag{10}$$

Continued

- Recall $g = 9.81\ meter/s^2$. With $m = 1000\ kg$. So, we have
$$\frac{dv}{dt} = 9.81 - \frac{1}{1000}\gamma v^2 \qquad (11)$$

Continued

Statement: Now suppose the vertical acceleration reduces to zero, when velocity $v(t) = 100\ meter/sec$. Compute the drag constant γ.

- We have, acceleration $\frac{dv}{dt} = 0$, when $v = 0$. Substituting in (11),
$$0 = 9.81 - \frac{1}{1000}\gamma(100^2).$$
- So, $\gamma = .981$ and the model is
$$\frac{dv}{dt} = 9.81 - \frac{.981}{1000}v^2 = \frac{.981}{1000}\left(10000 - v^2\right)$$

Continued

- We separate variables (see §2.3):

$$\int \frac{dv}{v^2 - 10000} = -\frac{.981}{1000} \int dt + c \quad \Longrightarrow$$

$$\int \frac{1}{200}\left(\frac{1}{v-100} - \frac{1}{v+100}\right) dv = -\frac{.981}{1000} + c \Longrightarrow$$

- $$\frac{1}{200} \ln\left|\frac{v-100}{v+100}\right| = -\frac{.981}{1000} + c \Longrightarrow$$

Continued

- $$\left|\frac{v-100}{v+100}\right| = Ce^{-.1962t} \quad \text{with} \qquad C = e^{200c} > 0$$
- So,
$$\frac{v-100}{v+100} = Ce^{-.1962t} \quad \text{with} \qquad -\infty < C < \infty$$
- Substituting $v(0) = 0$ we have $C = -1$

Continued

- So, the solution is given by

$$\frac{v-100}{v+100} = -e^{-.1962t} \Longrightarrow$$

$$v(t) = 100 - (v+100)e^{-.1962t}$$

- Next Level: Let $h = h(t)$ denote the vertical distance of the missile, from the point of ejection, at time t. So,

$$\frac{dh}{dt} = v = v(t) = 100 - (v+100)e^{-.1962t}$$

This equation can be solved to determine the height $h(t)$, of the missile, at time t.

Example 3: Concentration

Statement: A water reservoir contains 10^6 gallons of water. The water is not acceptable for human consumption, due the level of chemicals in the water. The concentration of this chemicals is .01 *gm/gallon*. Pure water is added to the pond at the rate of 1,000 gallons/h. The well mixed water drains out of the pond at the same rate . Model the total quantity of chemicals in the pond and determine the concentration of the chemicals after one year.

Continued

Solution:

- Let $Q(t)$ =quatity of the chemical in the pond, at time t.
- So, $Q(0) = .01 * 10^6 = 10^4$ gm.
- Part a): The rate of change

$$\frac{dQ}{dt} = -1000 * \frac{Q(t)}{10^6} = -\frac{Q(t)}{10^3}$$

- We can use the general solution solution (9) or rework it out. I will rework. We have

$$\int \frac{dQ}{Q} = -\int \frac{dt}{10^3} + c \quad c \text{ is a constant.}$$

$$\ln Q(t) = \frac{t}{10^3} + c.$$

Continued

$$\text{So,}\quad Q(t) = Ce^{-\frac{t}{10^3}} \quad C \geq 0 \ \text{ is a constant}$$

$$\text{Now,}\quad Q(0) = 10^4 \Longrightarrow 10^4 = C.$$

$$\text{So, the solution is}\quad Q(t) = 10^4 e^{-\frac{t}{10^3}}$$

Finally, after one year, $t = 365 * 24 = 8760$. So,

$$Q(1\ year) = Q(8760) = 10^4 e^{-\frac{8760}{10^3}} = 10^4 e^{-8.760}$$

So, the concentration is

$$= \frac{Q(1\ year)}{10^6} = \frac{10^4 e^{-8.760}}{10^6} \quad \text{per gallon. This is near zero.}$$

§1.3 Classification based on no of ind. variables

Two broad classifications of DEs are as follows:

- When a DE involves only a single independent variable x (or t), then it is called an Ordinary DE (also called ODE). Chapter 2, 3 would be on ODE.
- When a DE involves more than one independent variables $x_1, x_2, \ldots, x_n$, then it is called a Partial DE (also called PDE). PDEs will not be covered in this course.

Classification based on number of unknown variables

- There may only be one unknown dependent variable y, to be determined. As in linear algebra, only one DE (plus initial value) is needed to determine y.
- There may also be more than one unknown dependent variables $y_1, y_2, \ldots, y_m$, to be determined. As in linear algebra, a system of m (independent, in some sense) DE (plus initial values) are needed to determine $y_1, y_2, \ldots, y_m$. They will be called a System of DEs. We will consider such systems in chapter 7.

Based on Order of derivatives

- DEs can be classified based on highest order of derivation present. We will cover
  - First order DE (Chapter 2)
  - Second order DE (Chapter 3)

Linearity and non-linearity

- An ODE of order n is called linear, if it looks like

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = g(t)$$

This is also written as:

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots a_{n-1}(t)y^{(1)} + a_n(t)y = g(t)$$

$a_i(t), g(t)$ are functions of the independent variable t.