Math 220: Differential Equations
Homework and Problems

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Contents

Chapter 1

Introduction

1.1 Direction Fields

1. Draw a Direction Field for the DE

   $$y' = y \qquad \text{Here y = y(t) is a function of } t.$$

   Pick a suitable window, to show the behavior at $t = \infty$.

2. Draw a Direction Field for the DE

   $$y' = -y \qquad \text{Here y = y(t) is a function of } t.$$

   Pick a suitable window, to show the behavior at $t = \infty$.

3. Draw a Direction Field for the DE

   $$y' = y - 2 \qquad \text{Here y = y(t) is a function of } t.$$

   Pick a suitable window, to show the behavior at $t = \infty$.

4. Draw a Direction Field for the DE

   $$y' = -y - 2 \qquad \text{Here y = y(t) is a function of } t.$$

   Pick a suitable window, to show the behavior at $t = \infty$.

1.2 Solving Some ODEs

In this section, you can use Solution given in Equation 9 in § 1.2

1. Let $y = y(t)$ be a function of t. Solve the initial value problem

$$y' = y \quad y(0) = 100$$

2. Let $y = y(t)$ be a function of t. Solve the initial value problem

$$y' = -y \quad y(0) = 100$$

3. Let $y = y(t)$ be a function of t. Solve the initial value problem

$$y' = y - 2 \quad y(0) = 100$$

4. Let $y = y(t)$ be a function of t. Solve the initial value problem

$$y' = -y - 2 \quad y(0) = 100$$

Chapter 2

First Order ODEs

2.1 First Oder Linear ODEs

1. Consider the initial value problem (IVP):

$$\frac{dy}{dt} + 2y = e^{-2t} \qquad y(0) = y_0$$

   (a) Solve the IVP.

   (b) For the solution $y = y(t)$, find the $\lim_{t\to\infty} y(t)$.

   (c) For which values of y_0, the solution stabilizes at infinity?
   (We say $y(t)$ stabilizes at infinity, if $\lim_{t\to\infty} y(t)$ is finite.)

   (d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

Solution: The integrating factor

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int 2dt\right) = e^{2t}$$

So,

$$\frac{d}{dt}(y\mu(t)) = \mu(t)e^{-2t} = 1 \quad \Longrightarrow y\mu(t) = \int 1 \cdot dt + c = t + c \Longrightarrow$$

$$y = \frac{1}{\mu(t)}\left[t + c\right] = e^{-2t}\left[t + c\right], \quad y(0) = y_0 \Longrightarrow c = y_0$$

So, the solution is:

$$y = \frac{1}{\mu(t)}\left[t + c\right] = e^{-2t}\left[t + y_0\right]$$

We have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[e^{-2t}\left[t + y_0\right]\right] = 0$$

So, the solution stabilizes, for all values of y_0.

2. Consider the initial value problem (IVP):

$$t\frac{dy}{dt} + y = e^{-t} \qquad t > 0 \quad y(1) = y_1$$

   (a) Solve the IVP.

   (b) For the solution $y = y(t)$, find the $\lim_{t\to\infty} y(t)$.

   (c) For which values of y_1, the solution stabilizes at infinity?

   (d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

3. Consider the initial value problem (IVP):

$$t\frac{dy}{dt} + y = t^2 \qquad t > 0 \quad y(1) = y_1$$

   (a) Solve the IVP.

   (b) For the solution $y = y(t)$, find the $\lim_{t\to\infty} y(t)$.

   (c) For which values of y_1, the solution stabilizes at infinity?

   (d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

4. Consider the initial value problem (IVP):

$$\frac{dy}{dx} + \frac{1-2x}{x^2}y = 1, \qquad x > 0 \qquad y(1) = y_1$$

   (a) Solve the IVP.

(b) For the solution $y = y(x)$, find the $\lim_{t\to\infty} y(x)$.

(c) For which values of y_1, the solution stabilizes at infinity?

(d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

5. Consider the initial value problem (IVP)(Corrected):

$$\frac{dy}{dt} + y\sec^2 t = \tan t \sec^2 t \qquad y(0) = y_0$$

(a) Solve the IVP.

(b) Give an open interval in which the solution is valid.

6. Consider the initial value problem (IVP):

$$\frac{dy}{dx} + (xe^{-x})y = -4xe^{-x} \qquad y(-1) = \Omega$$

(a) Solve the IVP.

(b) For the solution $y = y(x)$, find the $\lim_{t\to\infty} y(x)$.

(c) For which values of Ω, the solution stabilizes at infinity?

(d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

7. Consider the initial value problem (IVP):

$$(1 - e^x)\frac{dy}{dx} + 3e^x y = e^x \qquad y(\ln 2) = 0 \qquad (\text{Assume x} > 0)$$

(a) Solve the IVP.

8. Consider the initial value problem (IVP):

$$\frac{dy}{dt} + \frac{3y}{t} = \frac{1}{t^2} \qquad t > 0, \quad y(1) = \Omega$$

(a) Solve the IVP.

(b) For the solution $y = y(x)$, find the $\lim_{t\to\infty} y(x)$.

(c) For which values of Ω, the solution stabilizes at infinity?

(d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

2.2 Separable ODEs

In this section, you can live your answer in implicit form, when it looks too complex to give an explicit solution.

1. Solve the ODE
$$\frac{dy}{dx} = \frac{1+y^2}{yx^2}$$

2. Solve the IVP
$$\frac{dy}{dx} = \frac{x}{y(1-x^2)}, \qquad y(0) = 4$$

3. Solve the IVP
$$\frac{dy}{dx} = y^2(x+2), \qquad y(0) = 1$$

4. Solve the IVP
$$\frac{dy}{dx} = y^2(2x+3x^2), \qquad y(1) = -1$$

5. Solve the ODE
$$\frac{dy}{dt} + y^2 \sec^2 t = 0$$

6. Solve the IVP
$$\cos y \frac{dy}{dt} + \sec^2 t = 0, \qquad y(0) = 0$$

7. Solve the ODE
$$\tan y \frac{dy}{dt} = 1$$

8. Consider the IVP:
$$\frac{dy}{dx} = \frac{2x+3}{1+y}, \quad y(0) = y_0$$

   (a) Solve the IVP, including the interval in which the solution is valid.

   (b) For the solution $y = y(x)$, find the $\lim_{t\to\infty} y(x)$.

   (c) For which values of y_0, the solution stabilizes at infinity?

2.3 Miscellaneous ODEs

2.3.1 Homogeneous Equations

1. Solve the Homogeneous ODE:

$$\frac{dy}{dx} = \frac{y^3 + 2x^2y}{x^3} \qquad \text{Assume} \quad x > 0$$

2. Solve the Homogeneous ODE:

$$\frac{dy}{dx} = \frac{5x - 3y}{3x + 5y} \qquad \text{Assume} \quad x > 0, y > 0$$

3. Solve the Homogeneous ODE:

$$\frac{dy}{dx} = \frac{y^3 + xy^2}{yx^2 - x^3} \qquad \text{Assume} \quad x > 0$$

2.3.2 Bernoulli's Equation

1. Solve the ODE (Bernoulli Equation):

$$\frac{dy}{dx} + \frac{xy}{1 + x^2} = x\sqrt{y}$$

2. Solve the ODE (Bernoulli Equation):

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

3. Solve the ODE (Bernoulli Equation):

$$\frac{dy}{dx} + \frac{y}{2} = \frac{(x - 1)y^3}{2}$$

4. Solve the ODE (Bernoulli Equation):

$$\frac{dy}{dx} + \frac{y}{x} = xy^2$$

2.4 Examples of ODE Models

No Homework

2.5 Existence and Uniqueness of Solutions

1. Consider the initial value problem (IVP)

$$\begin{cases} (t+1)(t-1)(t-2)\frac{dy}{dt} + e^{t^2}y & = \sin t^2 \\ y(-3) & = 1 \end{cases}$$

   Use Theorem 2.5.1 to determine the interval in which this IVP has unique solution. (Do not try to solve).

2. Consider the initial value problem (IVP)

$$\begin{cases} (t+1)(t-1)(t-2)\frac{dy}{dt} + e^{t^2}y & = \cos t^2 \\ y(.5) & = 1 \end{cases}$$

   Use Theorem 2.5.1 to determine the interval in which this IVP has unique solution. (Do not try to solve).

3. Consider the initial value problem (IVP)

$$\begin{cases} \cos t\frac{dy}{dt} + y & = \pi + t^2 \\ y(-\pi) & = 0 \end{cases}$$

   Use Theorem 2.5.1 to determine the interval in which this IVP has unique solution. (Do not try to solve).

4. Consider the initial value problem (IVP)

$$\begin{cases} \cos t\frac{dy}{dt} + y & = \pi + t^2 \\ y(3\pi) & = 0 \end{cases}$$

   Use Theorem 2.5.1 to determine the interval in which this IVP has unique solution. (Do not try to solve).

2.6 Equilibrium Solutions

1. Consider the ODE

$$\frac{dy}{dt} = yt \qquad -\infty < y(0) = y_0 < \infty$$

   (a) Determine the Equilibrium Solutions.
   (b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.
   (c) Establish the same analytically, as well.

2. Consider the ODE

$$\frac{dy}{dt} = -yt \qquad -\infty < y(0) = y_0 < \infty$$

   (a) Determine the Equilibrium Solutions.
   (b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.
   (c) Establish the same analytically, as well.

3. Consider the ODE

$$\frac{dy}{dt} = y \sin t \qquad -\infty < y(0) = y_0 < \infty$$

   (a) Determine the Equilibrium Solutions.
   (b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.
   (c) Establish the same analytically, as well.

4. Consider the ODE

$$\frac{dy}{dt} = y(2 + \sin t) \qquad -\infty < y(0) = y_0 < \infty$$

   (a) Determine the Equilibrium Solutions.
   (b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.

(c) Establish the same analytically, as well.

5. Consider the ODE

$$\frac{dy}{dt} = (y+1)(y-1)t \qquad\qquad -\infty < y(0) = y_0 < \infty$$

(a) Determine the Equilibrium Solutions.

(b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.

(c) Optionally, establish the same analytically, as well.

6. Consider the Autonomous ODE

$$\frac{dy}{dt} = (y+2)(y+1)y(y-1)(y-2) \qquad\qquad -\infty < y(0) = y_0 < \infty$$

(a) Determine the Equilibrium Solutions.

(b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.

(c) Avoid, analytic solution. It may be too time consuming.

2.7 Exact Equations

1. Prove the following ODEs are not Exact:

(a) Prove that the following ODE is not exact:

$$\sin(y) + \sin(xy)\frac{dy}{dx} = 0$$

(b) Prove that the following ODE is not exact:

$$\sin(x+y) + \sin(x)\frac{dy}{dx} = 0$$

(c) Prove that the following ODE is not exact:

$$e^{x+y} + xy\frac{dy}{dx} = 0$$

2. Prove that the ODE

$$(x^2 + xy^2 + 4x) + (x^2y - y^2 + y)\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

3. Prove that the ODE

$$(4x^3 + 3xy^2) + (4y^3 + 3x^2y)\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

4. Prove that the ODE

$$(1 + 6xy^2) + (1 + 6x^2y)\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

5. Prove that the ODE

$$\sin(x + y) + (1 + \sin(x + y))\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

6. Prove that the ODE

$$\cos x \cos y - \sin x \sin y \frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

7. Prove that the ODE

$$\left(\ln y + x^2\right) + \left(\frac{x}{y} + 2y\right)\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

8. Consider the ODE

$$(M_0(x) + M_1(x, y)) + (N_0(y) + N_1(x, y))\frac{dy}{dx} = 0$$

where $M_0(x)$ is a differentiable function of x, $N_0(y)$ is a differentiable function of y, and M_1, N_1 are differentiable functions of x, y. Prove:

$$\text{If} \qquad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}, \qquad \text{then the ODE is Exact.}$$

2.8 Numerical Solutions: Euler's Method

1. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = 3t^2 \\ y(1) = 1 \end{cases}$$

   (a) Compute the analytic solution $y = \varphi(t)$ and evaluate $\varphi(2)$.

   (b) Use Euler method to approximate $\varphi(2)$, with $h = .05$. Submit the Matlab or Excel output.

2. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = -y + t \\ y(0) = 2 \end{cases}$$

   (a) Compute the analytic solution $y = \varphi(t)$ and evaluate $\varphi(1)$.

   (b) Use Euler method to approximate $\varphi(1)$, with $h = .05$. Submit the Matlab or Excel output.

3. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = -y + t \\ y(0) = 1 \end{cases}$$

   (a) Compute the analytic solution $y = \varphi(t)$ and evaluate $\varphi(1)$.

   (b) Use Euler method to approximate $\varphi(1)$, with $h = .05$. Submit the Matlab or Excel output.

4. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = -y + \sin t \\ y(0) = \frac{1}{2} \end{cases}$$

   (a) Compute the analytic solution $y = \varphi(t)$ and evaluate $\varphi(\pi/2)$.

   (b) Use Euler method to approximate $\varphi(\pi/2)$, with $h = \frac{\pi}{40}$. Submit the Matlab or Excel output.

5. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = y^2 + t \\ y(0) = 1 \end{cases}$$

   (*We may not have discussed any method to solve this equation analytically.*)

(a) Use Euler method to approximate $\varphi(1)$, with $h = .05$. Submit the Matlab or Excel output.

Chapter 3

Second Order ODE

3.1 Introduction

1. Consider the general form of the Linear Homogenous ODE, of order two:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

Prove that the constant function $y = \varphi(t) = 0$ is a solution of this equation.

Remark. Note that the above problem is analogous to the following result in Linear Algebra:

Consider the homogeneous system of Linear Equations:

$$A\mathbf{x} = \mathbf{0} \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \quad \text{with } m \text{ rows,}$$

and A is a $m \times n$ matrix. Then, $\mathbf{x} = \mathbf{0}$ (with n rows) is a solution of this system.

3.2 Homogeneous Linear second order ODE, with Constant Coefficients

1. Give a general solution of the ODE
$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

2. Give a general solution of the ODE
$$9\frac{d^2y}{dx^2} - 9\frac{dy}{dx} - 4y = 0$$

3. Give a general solution of the ODE
$$9\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

4. Give a general solution of the ODE
$$4\frac{d^2y}{dx^2} - y = 0$$

5. Give a general solution of the ODE
$$\frac{d^2y}{dx^2} - \pi\frac{dy}{dt} - 2\pi^2 y = 0$$

6. Consider the IVP
$$\begin{cases} \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$
Solve the IVP $y = \varphi(t)$ and compute $\lim_{t\to\infty} \varphi(t)$.

7. Consider the IVP
$$\begin{cases} \frac{d^2y}{dx^2} - \pi^2 y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$
Solve the IVP $y = \varphi(t)$ and compute $\lim_{t\to\infty} \varphi(t)$.

8. Consider the IVP

$$\begin{cases} \frac{d^2y}{dx^2} - 9y = 0 \\ y(0) = \alpha \\ y'(0) = 1 \end{cases}$$

   (a) Solve the IVP $y = \varphi(t)$.

   (b) For what values of α the limit $\lim_{t\to\infty} \varphi(t)$ is finite?

9. Consider the IVP

$$\begin{cases} \frac{d^2y}{dx^2} + 10\frac{dy}{dt} = 0 \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$

   Solve the IVP $y = \varphi(t)$ and compute $\lim_{t\to\infty} \varphi(t)$.

10. Consider the IVP

$$\begin{cases} \frac{d^2y}{dx^2} - 10\frac{dy}{dt} = 0 \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$

   Solve the IVP $y = \varphi(t)$ and compute $\lim_{t\to\infty} \varphi(t)$.

11. Consider the IVP

$$\begin{cases} \frac{d^2y}{dx^2} - 10\frac{dy}{dt} + 21y = 0 \\ y(1) = 0 \\ y'(1) = 0 \end{cases}$$

   Solve the IVP.

3.3 Fundamental Set of Solutions

1. Consider the 2^{nd}-order linear homogeneous ODE:

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^{-\frac{1}{2}t} \\ y_2 = te^{-\frac{1}{2}t} \end{cases}$$

   Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

2. Consider the 2^{nd}-order linear homogeneous ODE:

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^{-\frac{1}{2}t} \\ y_2 = 7e^{-\frac{1}{2}t} \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

3. Consider the 2^{nd}-order linear homogeneous ODE:

$$4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 21y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^{-\frac{3}{2}t} \\ y_2 = e^{\frac{7}{2}t} \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

4. Consider the 2^{nd}-order linear homogeneous ODE:

$$4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 21y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^{\frac{7}{2}t} \\ y_2 = \pi e^{\frac{7}{2}t} \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

5. Consider the 2^{nd}-order linear homogeneous ODE:

$$\frac{d^2y}{dt^2} + 2y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = \cos(\sqrt{2}t) \\ y_2 = \sin(\sqrt{2}t) \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

6. Consider the 2^{nd}-order linear homogeneous ODE:

$$\frac{d^2y}{dt^2} + 2y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = \cos(\sqrt{2}t) \\ y_2 = 3\cos(\sqrt{2}t) \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

7. Consider the 2^{nd}-order linear homogeneous ODE:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^t \cos(2t) \\ y_2 = e^t \sin(2t) \end{cases}$$

(*Yes, here $y_1 = y_2$, not a typo.*) Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

8. Consider the 2^{nd}-order linear homogeneous ODE:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^t \cos(2t) \\ y_2 = e^t \cos(2t) \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

9. Consider the general form of the 2^{nd}-order linear homogeneous ODE:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad a \neq 0$$

with constant coefficients $a, b, c \in \mathbb{R}$. Let $y = y_1, y = y_2$ be two solution of the ODE. Use Abel's Theorem, to compute the Wronskian $W(y_1, y_2)$, up to a constant multiplier.

Hint: See the same Lemma in §3.3 and reproduce!

3.4 Repeated roots of the CE

1. Solve IVP

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(0) = 3 \\ y'(0) = -1 \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

2. We change the initial condition in the above problem (**??**):

Solve IVP

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(1) = 3e^{-\frac{1}{2}} \\ y'(1) = -e^{-\frac{1}{2}} \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

3. Solve IVP

$$25\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(0) = 5 \\ y'(0) = -1 \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

4. We change the initial condition in (5).

   Solve IVP

$$25\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(1) = 5e^{\frac{1}{5}} \\ y'(1) = -e^{\frac{1}{5}} \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

5. Solve IVP

$$25\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(0) = 5 \\ y'(0) = -1 \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

6. Solve IVP

$$\frac{d^2y}{dt^2} + 14\frac{dy}{dt} + 49y = 0, \qquad \begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

3.5 Complex roots of the CE

1. Solve IVP

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = 0, \qquad \begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases}$$

   Also describe the nature of the solution, as $t \to \infty$.

2. Solve IVP

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = 0, \qquad \begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases}$$

   Also describe the nature of the solution, as $t \to \infty$.

3. Solve IVP

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + (4 + \pi^2)y = 0, \qquad \begin{cases} y(1) = -e^2 \\ y'(1) = e^2 \end{cases}$$

   Also describe the nature of the solution, as $t \to \infty$.

4. Solve IVP

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + (4+\pi^2)y = 0, \qquad \begin{cases} y(1) = -e^{-2} \\ y'(1) = e^{-2} \end{cases}$$

   Also describe the nature of the solution, as $t \to \infty$.

5. Solve IVP

$$\frac{d^2y}{dt^2} + 49y = 0, \qquad \begin{cases} y(\pi) = -1 \\ y'(\pi) = 7 \end{cases}$$

   Also describe the nature of the solution, as $t \to \infty$.

6. Solve IVP

$$\frac{d^2y}{dt^2} + 4\pi^2 y = 0, \qquad \begin{cases} y(1) = -1 \\ y'(1) = 2 \end{cases}$$

   Also describe the nature of the solution, as $t \to \infty$.

3.6 Nonhomogeneous ODE: Method of Variations of Parameters

For some of the problems, the integration Formula A.1.1, would be helpful.

1. Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = e^{5t}$$

2. Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = e^{t}$$

3. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = e^{10t}$$

4. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = e^{-\sqrt{5}t}$$

5. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + (4 + \pi^2)y = 1$$

6. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 49y = 2\sin 7t$$

7. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = \cos 5t$$

3.6.1 More Problems

Do not submit the following of the problems in this subsection. The following problems are variations of the above problems, where right hand side $g(t)$ would be different. Majority of the steps would be same as above, while the integrations would be more involved.

1. The following are variations of Problem 1.

   (a) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = (1 + t + t^2)e^{5t}$$

   (b) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = e^{5t}\cos 3t$$

(c) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = (1+t+t^2)\cos 3t$$

2. The following are variations of Problem 2.

(a) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = (1+t-t^2)e^t$$

(b) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = e^t \sin 2t$$

(c) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = (1+t+t^2)\sin 2t$$

3. The following are variations of Problem 3.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = (1+t-t^2)e^{10t}$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = e^{10t}\cos 3t$$

(c) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = (1+t+t^2)\cos 3t$$

4. The following are variations of Problem 4.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = (1+t+t^2)e^{-\sqrt{5}t}$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = e^{-\sqrt{5}t}\sin 2t$$

(c) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = (1+t+t^2)\sin 2t$$

5. The following are variations of Problem 5.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + (4+\pi^2)y = 1+t+t^2$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + (4+\pi^2)y = (1+t+t^2)\sin 2t$$

6. The following are variations of Problem 6.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 49y = 2t^2\sin 7t$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 49y = 2e^{3t}\sin 7t$$

7. The following are variations of Problem 7.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = (1+t+t^2)\cos 5t$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = e^{t}\cos 5t$$

3.7 Method of Undetermined Coefficients

No new problems will be assigned in the section. One can try to solve any of the problems in Section 3.6 or Section 3.6.1, using this method of undetermined coefficients.

3.8 Elements of Particle Dynamics

We would not assign any Homework on this section. Problems are essentially covered by what we did in § 3.5, 3.6, 3.6.1, 3.7.

The reason for this departure from the customary practice is two fold. The problem sets on this topic in the literature appears a little artificial, to me. Some of problems are, essentially same as those in § 3.5, 3.6, 3.6.1, 3.7, encased within a story on Mechanics. Other set of problems, ask to compute Amplitude, Periodicity etc., which may belong in the Mechanics classes.

Chapter 4

Higher Order ODE

4.1 General Overview of Theory

No Homework

4.2 Linear Homogeneous ODE with constant coefficients

1. Give a general solution of the Homogeneous ODE

$$\frac{d^3y}{dt^3} - y = 0$$

2. Give a general solution of the Homogeneous ODE

$$\frac{d^6y}{dt^6} - y = 0$$

3. Give a general solution of the Homogeneous ODE

$$\frac{d^4y}{dt^4} - \pi^4 y = 0$$

4. Give a general solution of the Homogeneous ODE

$$\frac{d^4y}{dt^4} - 4\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 16\frac{dy}{dt} - 32y = 0$$

4.3 Nonhomogeneous Linear ODE

No Homework

Chapter 5

System of 1^{st}-Order Linear ODE

5.1 Introduction

No Homework

5.2 Algebra Of Matrices

No Homework

5.3 Linear Systems and Eigen Values

Definition 5.3.1. Let A be a square matrix of order n, with real entries, and $\lambda \in \mathbb{R}$ be a Eigen value of A. Then, the Eigen Space $E(\lambda)$ is defined to be the set of all Eigen Vectors corresponding to λ, together with the zero vector. So,

$$E(\lambda) = \{\mathbf{x} \in \mathbb{R}^n : (A - \lambda I_n)\mathbf{x} = \mathbf{0}\}$$

Note, $E(\lambda)$ is a subspace of $\mathbb{R}^n$.

If $\lambda \in \mathbb{C}$ is a complex Eigen value of A, then the Eigen Space $E(\lambda)$ is defined to be

$$E(\lambda) = \{\mathbf{x} \in \mathbb{C}^n : (A - \lambda I_n)\mathbf{x} = \mathbf{0}\}$$

1. Let
$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
   (a) Write down the characteristic equation of A
   (b) Find all the eigen values of A.
   (c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.
2. Let
$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$
   (a) Write down the characteristic equation of A
   (b) Find all the eigen values of A.
   (c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.
3. Let
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$
   (a) Write down the characteristic equation of A
   (b) Find all the eigen values of A.
   (c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.
4. Let
$$A = \begin{pmatrix} 1 & 2 & -6 \\ -2 & 5 & -6 \\ -2 & 2 & -3 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

5. Let

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 4 & 1 & -2 \\ -4 & 2 & 5 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

6. Let

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

7. Let

$$A = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

8. Let

$$A = \begin{pmatrix} 4 & 3 & -5 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

5.4 The Theoretical Foundation

No Homework

5.5 Homogeneous Systems with Constant Coefficients

Consider homogeneous systems $\mathbf{y}' = A\mathbf{y}$, where A a constant matrix, of size $n \times n$. This section deals with problems, such that the roots of the characteristic Equation $|A - \lambda I| = 0$ are real and distinct. Consequently, the corresponding eigen vectors would be linearly independent, which lead to a Fundamental Set of Solutions.

1. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 6 & 3 \\ 10 & 5 \end{pmatrix} \mathbf{y}$$

2. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} -1 & 3 \\ 4 & 3 \end{pmatrix} \mathbf{y}$$

3. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 3 & 8 \\ 2 & -3 \end{pmatrix} \mathbf{y}$$

4. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{y}$$

5. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix} \mathbf{y}$$

5.6 Complex Eigenvalues

Consider homogeneous systems $\mathbf{y}' = A\mathbf{y}$, where A a constant matrix, of size $n \times n$. The section deals with complex values of the characteristic Equation $|A - \lambda I| = 0$. Before we proceed, recall from §5.6, corresponding a pari of conjugate eigen value $r = \lambda \pm i\mu$, and eigen vector $\xi = \mathbf{a} + i\mathbf{b}$, two solutions

$$\begin{cases} \mathbf{u} = e^{\lambda t}\left(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t\right) \\ \mathbf{v} = e^{\lambda t}\left(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t\right) \end{cases} \tag{5.1}$$

1. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 2 & -1 \\ 13 & -2 \end{pmatrix} \mathbf{y}$$

2. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix} \mathbf{y}$$

3. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} -1 & \pi \\ -\pi & -1 \end{pmatrix} \mathbf{y}$$

4. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} -1 & 7 & 0 \\ -7 & -1 & 0 \\ 3 & 4 & 4 \end{pmatrix} \mathbf{y}$$

5. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 5 & 0 & 9 \\ 0 & \pi & 0 \\ -4 & 0 & -5 \end{pmatrix} \mathbf{y}$$

5.7 Repeated Egenvalues

1. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} \mathbf{y}$$

2. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} 3 & -5 \\ 5 & -7 \end{pmatrix} \mathbf{y}$$

3. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} \pi & -\pi \\ \pi & 3\pi \end{pmatrix} \mathbf{y}$$

4. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \mathbf{y}$$

   Help: There would be two linearly independent eigen vector, for the double eigen value.

5. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ -1 & -1 & 1 \end{pmatrix} \mathbf{y}$$

   Help: There would be Only ONE linearly independent eigen vector, for the double eigen value.

5.8 Nonhomogeneous Linear Systems

For the purpose of this course, we consider problems in this sections, so that the respective eigen values are real and distinct.
(Please double check for possible numerical errors.)

1. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$$

   Make sure to show the following steps:

   (a) Compute the matrix T of eigen vectors.
   (b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.
   (c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.
   (d) Write down a general solution for $\mathbf{y}$

2. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} t \\ e^{-2t} \end{pmatrix}$$

   Make sure to show the following steps:

   (a) Compute the matrix T of eigen vectors.
   (b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.
   (c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.
   (d) Write down a general solution for $\mathbf{y}$

3. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} t \\ 2t \end{pmatrix}$$

   Make sure to show the following steps:

   (a) Compute the matrix T of eigen vectors.
   (b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

4. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} \cos t \\ \cos t \end{pmatrix}$$

Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

5. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -e^{2t} \\ -e^{2t} \\ e^{2t} \end{pmatrix}$$

Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

6. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ 2 \\ 2t \end{pmatrix}$$

Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

Chapter 6

The Laplace Transform

6.1 Definition of Laplace Transform

1. Compute the Laplace Transform of the function $f(t) = t$, from the definition. (That means, do not use the Charts.)

2. Compute the Laplace Transform of the function $f(t) = \sin 3t$, from the definition. (That means, do not use the Charts.)

3. Compute the Laplace Transform of the function $f(t) = \cos 2t$, from the definition. (That means, do not use the Charts.)

4. Compute the Laplace Transform of the function

$$f(t) = \begin{cases} \sin \pi t & if\ t \leq 1 \\ 0 & if\ 1 < t \end{cases}$$

   from the definition. (That means, do not use the Charts.)

5. Compute the Laplace Transform of the function

$$f(t) = \begin{cases} \cos \pi t & if\ t \leq 1 \\ -1 & if\ 1 < t \end{cases}$$

   from the definition. (That means, do not use the Charts.)

6. (Do not Submit This one.) Compute the Laplace Transform of the function $f(t) = t\sin\pi t$, from the definition. (That means, do not use the Charts.)

6.2 Solutions of Initial Value Problems

Use the Laplace Transform Charts available in the internet, to solve the problems in this section.

1. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 10y = 0, \quad \begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$$

2. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0, \quad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

3. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 16y = e^{-4t}, \quad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

4. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 5y = \cos 2t, \quad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

6.3 Step Functions and Dirac Delta

1. Compute the Laplace Transform of the function

$$u_2(t) = \begin{cases} 0 & if\ t < 2 \\ 1 & if\ 2 \leq t \end{cases}$$

2. Compute the Laplace Transform of the function

$$f(t) = \begin{cases} 0 & if\ t < 2 \\ 1 & if\ 2 \leq t < 3 \\ 0 & if\ 3 \leq t \end{cases}$$

3. Compute the Laplace Transform of the function

$$f(t) := d_{.01}(t-2) = \begin{cases} 0 & if\ t < 1.99 \\ 100 & if\ 1.99 \leq t < 2.01 \\ 0 & if\ 3 \leq t \end{cases}$$

   (It is the same function $d_\tau(t - t_0)$ in the notes.)

4. Compute the Laplace Transform of the Dirac Delta $\delta(t-2)$. (You can use the formula).

6.4 Systems with Discontinuous Functions

Some of the problems in § **??** could fall in this section.

1. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2} + 9y = u_\pi, \quad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Appendix A

Appendix

A.1 A Formula

Lemma A.1.1.

$$\int e^{\lambda t}\cos\mu t dt = e^{\lambda t}\frac{\mu\sin\mu t + \lambda\cos\mu t}{\lambda^2+\mu^2}$$

$$\int e^{\lambda t}\sin\mu t dt = e^{\lambda t}\frac{\lambda\sin\mu t - \mu\cos\mu t}{\lambda^2+\mu^2}$$

Proof.

$$I = \int e^{\lambda t}\cos\mu t dt = \frac{1}{\mu}\int e^{\lambda t} d\sin\mu t = \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t - \lambda\int \sin\mu t e^{\lambda t} dt\right)$$

$$= \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t + \frac{\lambda}{\mu}\int e^{\lambda t} d\cos\mu t\right)$$

$$= \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t + \frac{\lambda}{\mu}\left(e^{\lambda t}\cos\mu t - \lambda\int e^{\lambda t}\cos\mu t dt\right)\right)$$

$$= \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t + \frac{\lambda}{\mu}\left(e^{\lambda t}\cos\mu t - \lambda I\right)\right)$$

$$\left(\frac{\lambda^2+\mu^2}{\mu^2}\right) I = \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t + \frac{\lambda}{\mu}\left(e^{\lambda t}\cos\mu t\right)\right) = e^{\lambda t}\frac{\mu\sin\mu t + \lambda\cos\mu t}{\mu^2}$$

So,

$$I = e^{\lambda t}\frac{\mu\sin\mu t + \lambda\cos\mu t}{\lambda^2+\mu^2}$$

Now,

$$J := \int e^{\lambda t}\sin\mu t dt = -\frac{1}{\mu}\int e^{\lambda t} d\cos\mu t = -\frac{1}{\mu}\left(e^{\lambda t}\cos\mu t - \lambda\int e^{\lambda t}\cos\mu t dt\right)$$

$$= -\frac{1}{\mu}\left(e^{\lambda t}\cos\mu t - \lambda e^{\lambda t}\frac{\mu\sin\mu t + \lambda\cos\mu t}{\lambda^2+\mu^2}\right)$$

$$= -\frac{1}{\mu}e^{\lambda t}\left(\cos\mu t - \frac{\lambda\mu\sin\mu t + \lambda^2\cos\mu t}{\lambda^2+\mu^2}\right)$$

$$= -\frac{1}{\mu}e^{\lambda t}\left(\frac{-\lambda\mu\sin\mu t + \mu^2\cos\mu t}{\lambda^2+\mu^2}\right) = e^{\lambda t}\frac{\lambda\sin\mu t - \mu\cos\mu t}{\lambda^2+\mu^2}$$