Chapter 3:Second Order ODE §3.3 Fundamental Set of Solutions of Homogeneous LSODEs

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SODEs

- Recall, second order ODE (SODE) has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \tag{1}$$

This is also written as

$$y" = f(t, y, y')$$

LSODE

- A linear second order ODE (LSODE), is often written as:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{2}$$

This is also written as

$$y'' + p(t)y' + q(t)y = g(t)$$

where $p(t), q(t), g(t)$ are functions of t.

- Another form of second order ODE (2) is:

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t) \qquad (3)$$

where $P(t), Q(t), R(t), G(t)$ are functions of t. This is also written as

$$P(t)y" + Q(t)y' + R(t)y = G(t)$$

Homogeneous Equations

- The ODEs (2, 3) would be called homogeneous, if $g(t) = 0$ or $G(t) = 0$. So, it looks like:

$$y'' + p(t)y' + q(t)y = 0 \quad \textit{or} \quad P(t)y'' + Q(t)y' + R(t)y = 0 \tag{4}$$

- (The Trivial Solution): The constant $y = 0$ is a solution of any such homogeneous equation (4). (*This property is analogous to that of system of homogeneous linear equations* $A\mathbf{x} = \mathbf{0}$, *in algebra, where* $\mathbf{x} = \mathbf{0}$ *is the trivial solutuon.*)
- In previous section, we considered LSODEs with constant coefficients.

Main Point

- Perhaps, the main point of this section is Theorem 3.3.3 regarding Fundamental Set of Solutions.
- We also State Existence and Uniqueness Theorem 3.3.2, for Linear Homogeneous ODE, of order two.

Derivative as an operator

- It is helpful think of derivative $D = \frac{d}{dt}$ as an operator.
- Given any differentiable function $\varphi(t)$, $D = \frac{d}{dt}$ operates on $\varphi(t)$ and produces the derivative $D(\varphi) = \frac{d\varphi}{dt}$.
- D sends

$$\varphi \quad \mapsto \quad D(\varphi) = \frac{d\varphi}{dt}.$$

- We extend this idea of "operators" in the next frame, in the context of linear second order ODE (LSODE).

Differential Operators

- Suppose $p(t), q(t)$ are two continuous functions on an open interval $I = (\alpha, \beta)$, which means: $\alpha < t < \beta$. We define a differential operator $\mathcal{L}$, which operates on all twice differentiable functions $\varphi(t)$ on I as follows:

$$\mathcal{L}(\varphi) := \frac{d^2\varphi}{dt^2} + p\frac{d\varphi}{dt} + q\varphi \qquad (5)$$

  This is also written as $\mathcal{L}(\varphi) := \varphi'' + p\varphi' + q\varphi$.
- We also write $\mathcal{L} = D^2 + pD + q$ where $D = \frac{d}{dt}$.

Continued

- Such operators are like "functions". Given a twice differentiable functions φ, the "operation" $\mathcal{L}$ operates on φ and produces a new function $\mathcal{L}(\varphi) := \varphi'' + p\varphi' + q\varphi$.

$$\mathcal{L} \quad \textit{associates} \quad \varphi \quad \mapsto \quad \mathcal{L}(\varphi) := \varphi'' + p\varphi' + q\varphi$$

Continued

- Example: $\mathcal{L} = D^2 + 2e^t D + \sqrt{t}$ is a differential operator.
  - When it operates on $\varphi(t) = t^3 + \sin t$, then $\mathcal{L}(t^3 + \sin t)$

$$= D^2(t^3 + \sin t) + 2e^t D(t^3 + \sin t) + \sqrt{t}(t^3 + \sin t)$$

$$= (6t - sint) + (3t^3 + \cos t) + \sqrt{t}(t^3 + \sin t)$$

- Example: $\mathcal{L} = D^2 + \sin(2t)D + \ln t$ is a differential operator.
  - When it operates on $\varphi(t) = e^{2t}$, then

$$\mathcal{L}(e^{2t}) = D^2(e^{2t}) + \sin(2t)D(e^{2t}) + \ln t(e^{2t})$$

$$= 4e^{2t} + \sin(2t)(2e^{2t}) + \ln t(e^{2t})$$

Properties and Plan

- Properties: Let $\mathcal{L} = D^2 + pD + q$. Then, $\mathcal{L}$ is a Linear Operator, in the following sense:
  - $\mathcal{L}$ is Linear, in the sense, for any two differentiable function $y = \varphi_1(t)$, $y = \varphi_2(t)$, and for $a \in \mathbb{R}$, we have
$$\begin{cases} \mathcal{L}(\varphi_1 + \varphi_2) = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2) \\ \mathcal{L}(a\varphi_1) = a\mathcal{L}(\varphi_1) \end{cases}$$
  - Putting them together for scalars $a, b \in \mathbb{R}$ we have:
$$\mathcal{L}(a\varphi_1 + b\varphi_2) = a\mathcal{L}(\varphi_1) + b\mathcal{L}(\varphi_2) \qquad (6)$$
- Plan: Get used to the idea (jargon) of such operators $\mathcal{L}$. Use this jargon to express LSODEs.

Consequence of the Properties

Linear Combination of two solutions:

- Theorem 3.3.1: Suppose $\mathcal{L} = D^2 + pD + q$ is a differential operator. Consider the homogeneous LSODE $\mathcal{L}(y) = 0$. Let $y = \varphi_1(t), y = \varphi_2(t)$ be two solutions of this ODE. Then, for any constants $c_1, c_2 \in \mathbb{R}$, the linear combination $y = c_1\varphi_1(t) + c_2\varphi_2(t)$ is also a solution of this equation.
  Proof. By (6) we have

$$\mathcal{L}(c_1\varphi_1 + c_2\varphi_2) = c_1\mathcal{L}(\varphi_1) + c_2\mathcal{L}(\varphi_2) = c_1 * 0 + c_2 * 0 = 0.$$

The proof is complete. ■

Existence and Uniqueness

- Given any equation (in math or life), existence of a solution is not guaranteed. If and when, there is a solution, there is no guarantee that the solution would be unique. We seek conditions, under which, there are such guarantees.
- In §2.5 we dealt with these questions for first order Linear ODEs.In complete analogy to the Existence and Uniqueness Theorem (2.5.1) for 1^{st}-order Linear IVPs, in the next frame, we state the Existence and Uniqueness Theorem for 2^{nd}-order IVPs.

The Existence and Uniqueness Theorem

- Theorem 3.3.2. Consider the 2^{nd}-order Linear IVP

$$\begin{cases} y'' + p(t)y' + q(t)y & = g(t) \\ y(t_0) & = y_0 \\ y'(t_0) & = y'_0 \end{cases} \tag{7}$$

Assume $p(t), q(t), g(t)$ are continuous on an open interval $I : \alpha < t < \beta$ and t_0 in I. Then,

  - The IVP (7) has a solution $y = \varphi(t)$.
  - The domain of $y = \varphi(t)$ is I,
  - The solution $y = \varphi(t)$ is unique, on I.

Remark

Since the above Existence and Uniqueness Theorem (3.3.2) is completely analogous to the corresponding theorem for First Order ODE (Theorem 2.5.1), we would skip any further discussion on this Theorem.

Further Goals

Consider a 2^{nd}-order Linear Homogeneous ODE, on an open interval $I : \alpha < t < \beta$:

$$\begin{cases} \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \qquad or \\ P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \end{cases} \tag{8}$$

Assume $p(t), q(t)$ etc. are continuous on I. Write it (8) as:

$$\mathcal{L}(y) = 0 \quad \text{where} \quad \mathcal{L} = \begin{cases} \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t) \qquad \text{OR} \\ P(t)\frac{d^2}{dt^2} + Q(t)\frac{d}{dt} + R(t) \end{cases} \tag{9}$$

Continued

We know,

- The ODE (9) has the trivial solution $y = 0$.
- Also, if $y = \varphi_1(t), y = \varphi_2(t)$ solutions of (9), then any constant linear combination

$$y = a\varphi_1 + b\varphi_2 \quad \text{is a solutions of (9).}$$

Continued

- Question: Suppose $y = \varphi_1(t), y = \varphi_2(t)$ are two solutions of the ODE (9). Suppose, $y = \varphi(t)$ is any other solution of (9). Question is, can we write φ as a constant linear combinations of φ_1 and φ_2?

  We investigate, under what conditions on $y = \varphi_1(t), y = \varphi_2(t)$, such is the case?

Definition: The Fundamental Set

Definition: Suppose $y = \varphi_1(t), y = \varphi_2(t)$ are two solutions of the ODE (9). We say that $y = \varphi_1(t), y = \varphi_2(t)$ form a Fundamental Set of solutions, if any other solution $y = \varphi(t)$ can be written as a constant linear combination of $y = \varphi_1(t), y = \varphi_2(t)$. That means, if

$$y = \varphi(t) = a\varphi_1(t) + b\varphi_2(t) \quad \forall\ t \in I \quad \text{for some } a, b \in \mathbb{R}$$

(*Now, we investigate, when $y = \varphi_1(t), y = \varphi_2(t)$ would be a Fundamental Set of solutions.*)

Wronskian

Definition. Let $y = \varphi_1(t), y = \varphi_2(t)$ be two differentiable functions on an open interval $I : \alpha < t < \beta$. The Wronskian $W(t)$, of these two functions is defined to be the function:

$$W(t) = \begin{vmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{vmatrix} \qquad t \in I \qquad (10)$$

Sometimes,to indicate its dependence on φ_1, φ_2, $W(t)$ is denoted by

$$W(\varphi_1, \varphi_2)(t) := W(t)$$

.

The (Wronskian) Theorem 3.3.3

Theorem 3.3.3 Consider the 2^{nd}-order Linear ODE (9). Fix $t_0 \in I$. Let $y = \varphi_1(t), y = \varphi_2(t)$ be two solutions of (9). Let $W(t)$ denote the Wronskian of $y = \varphi_1(t), y = \varphi_2(t)$. Then, the following three conditions are equivalent:

(1) $W(t) \neq 0$ for all $t \in I$.

(2) $W(t_0) \neq 0$.

(3) $y = \varphi_1(t), y = \varphi_2(t)$ form a Fundamental set (pair) of Solutions of (9).

The Proof.

(1) $\Longrightarrow$ (2) is obvious. To prove (2) $\Longrightarrow$ (3), let $y = \varphi(t)$ be a solution of (9). We need to prove that $\varphi = c_1\varphi_1 + c_2\varphi_2$. Write $y_0 = \varphi(t_0)$, $y_0' = \varphi'(t_0)$. Consider the system of two linear equations

$$\begin{pmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

We have

$$W(t_0) = \begin{vmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{vmatrix} \neq 0$$

Proof: Continued

By Cramer's Rule, in Linear Algebra (Math 290), the above system has a unique solution, given by

$$c_1 = \frac{\begin{vmatrix} y_0 & \varphi_2(t_0) \\ y_0' & \varphi_2'(t_0) \end{vmatrix}}{W(t_0)}, \qquad c_2 = \frac{\begin{vmatrix} \varphi_1(t_0) & y_0 \\ \varphi_1'(t_0) & y_0' \end{vmatrix}}{W(t_0)} \qquad (11)$$

With c_1, c_2, as in (11), let

$$\psi(t) = c_1\varphi_1 + c_2\varphi_2$$

Proof: Continued

- Both ψ and φ are solutions of (11).
- They are both solutions of the IVP:

$$\begin{aligned} \mathcal{L}(y) = y'' + p(t)y' + q(t)y &= 0 \\ y(t_0) &= y_0 \\ y'(t_0) &= y'_0 \end{aligned}$$

- By uniqueness part of Theorem 3.3.2, $\varphi = \psi = c_1\varphi_1 + c_2\varphi_2$.

So, (3) is established. That means φ_1, φ_2 forms a Fundamental set of solutions.

Proof: Continued

To prove (3) $\Longrightarrow$ (1), assume $W(\tau_0) = 0$ for some $\tau_0 \in I$.
Claim: There are choices of real numbers y_0, y_0', to be determined, such that the system

$$\begin{pmatrix} \varphi_1(\tau_0) & \varphi_2(\tau_0) \\ \varphi_1'(\tau_0) & \varphi_2'(\tau_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \qquad (12)$$

has no solutions. For notational convenience, denote

$$C = \begin{pmatrix} \varphi_1(\tau_0) & \varphi_2(\tau_0) \\ \varphi_1'(\tau_0) & \varphi_2'(\tau_0) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We have $$Adj(C)C = \begin{pmatrix} W(\tau_0) & 0 \\ 0 & W(\tau_0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Continued

Multiplying Equation 12 by $Adj(C)$ we have,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = Adj(C) \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & d \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

$$= \begin{pmatrix} dy_0 - by_0' \\ -cy_0 + dy_0'' \end{pmatrix}$$

$$\text{Chose} \quad y_0, y_1 \quad \ni \quad \begin{pmatrix} dy_0 - by_0' \\ -cy_0 + dy_0'' \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (13)$$

With this choice, the linear system (12) would not have any solution.

Continued

With the above choice (13) of y_0, y_0', consider the IVP

$$\begin{aligned} \mathcal{L}(y) = y'' + p(t)y' + q(t)y &= 0 \\ y(\tau_0) &= y_0 \\ y'(\tau_0) &= y_0' \end{aligned}$$

By existence part of Theorem 3.3.2, this IVP has a unique solution $y = \varphi(t)$. This solution $y = \varphi(t)$ cannot be written as a linear combination $\varphi = c_1\varphi_1 + c_2\varphi_2$. Because, such c_1, c_2 must be a solution of the linear system (12).
The Proof of the Theorem is complete.

Example 1

Compute the Wronskian of $y_1 = \sin t, y_2 = \cos t$.

Solution:

- The derivatives $\begin{cases} y_1' = \cos t \\ y_2' = -\sin t \end{cases}$
- The Wronskian:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1' & y_2'(t) \end{vmatrix} = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -1$$

Example 2

Compute the Wronskian of $y_1 = e^{2t}, y_2 = e^{-2t}$.

Solution:

- The derivatives $\begin{cases} y_1' = 2e^{2t} \\ y_2' = -2e^{-2t} \end{cases}$
- The Wronskian:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1' & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{vmatrix} = -4$$

Example 3

Consider the 2^{nd}-order ODE

$$\frac{d^2y}{dt} - 2\frac{dy}{dt} - 3y = 0$$

- Compute a pair of solutions, as in §3.2.
- Compute the Wronskian of this pair.
- Use Theorem 3.3.3 to conclude that this pair is a Fundamental set of solutions.

Solution

- The CE: $r^2 - 2r - 3 = 0$. So, $r_1 = -1$, $r_2 = 3$
- So,
$$y_1 = e^{r_1 t} = e^{-t}, \qquad y_2 = e^{r_2 t} = e^{3t}$$
are two solutions.
- The Wronskian:
$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{vmatrix} = 4e^{2t}$$

Continued

- Finally: Since $W(t) = 4e^{2t} \neq 0$, this pair of solutions $y_1 = e^{-t}, y_2 = e^{3t}$ form a Fundamental set of solution.

The Example 3 is a particular case of the following Lemma:

Lemma 3.3.4 Consider a 2^{nd}-order ODE with constant coefficients:

$$a\frac{d^2y}{dt} + b\frac{dy}{dt} + cy = 0$$

Suppose the Characteristic Equation $ar^2 + br + c = 0$ has two real roots $r = r_1, r_2$, with $r_1 \neq r_2$. Then,

$$\left\{ \begin{array}{l} y_1 = e^{r_1 t} \\ y_2 = e^{r_2 t} \end{array} \right. \quad \text{form a Fundamental Set of Solutions}$$

of the ODE.

Proof.

The Wronskian:

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0$$

By Theorem 3.3.3 they $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$ form a Fundamental Set of Solutions. The proof is complete.

Remark: Other Two Cases

Remark. We would see in the next two sections that, Lemma 3.3.4 remains valid, even when the Characteristic Equation $ar^2 + br + c = 0$ of the ODE

$$a\frac{d^2y}{dt} + b\frac{dy}{dt} + cy = 0$$

has two repeated real root or two conjugate complex roots. Next few problems are "warm up" for the same.

Example 4

Consider the ODE $\frac{d^2y}{dt^2} + 9y = 0$. Consider the functions $y_1(t) = \cos 3t$, $y_2(t) = \sin 3t$. (1) Verify, if y_1, y_2 are solutions of this DE, (2) If yes, do they form a fundamental set of solutions?

- Check, if $y_1(t) = \cos 3t$ is a solution.

$$y_1' = -3\sin 3t, y_1" = -9\cos 3t \Longrightarrow$$

$$y_1" + 4y_1 = -9\cos 3t + 4\cos 3t = 0$$

So, $y_1(t) = \cos 2t$ is a solution of this ODE. Similarly, so is $y_2(t) = \sin 3t$.

Continued

- The Wronskian:

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 3\cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3$$

- Finally: In deed, $W(t) = 3 \neq 0$. By (3.3.3), $y_1 = \cos 3t$, $y_2 = \sin 3t$ form a fundamental set of solutions.
- **Remark.** The CE $r^2 + 9 = 0$ had Complex roots, to be dealt with in §3.5.

Example 5

Consider the ODE

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = 0. \quad \text{Let} \quad \begin{cases} y_1 = e^t \sin t \\ y_2 = e^t \cos t \end{cases}$$

(1) Check, y_1, y_2 are solutions of this ODE. (2) Prove they form a fundamental set of solutions.

Solution:

We have

$$\frac{dy_1}{dt} = e^t(\sin t + \cos t), \quad \frac{d^2y_1}{dt^2} = 2e^t \cos t$$

So,

$$\frac{d^2y_1}{dt^2} - 2\frac{dy_1}{dt} + 2y_1 = 2e^t \cos t - 2(e^t(\sin t + \cos t)) + 2e^t \sin t = 0$$

So, y_1 is a solution. Likewise, y_2 is a solution.

Continued:

$$\text{Compute} \quad \frac{dy_2}{dt} = e^t(\cos - \sin t),$$

So, the Wronskian: $W(t) =$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos - \sin t) \end{vmatrix} = -e^{2t} \neq 0$$

By Theorem 3.3.3 y_1, y_2 form a fundamental set of solutions.

Remark. The CE $r^2 - 2r + 2 = 0$ had Complex roots, to be dealt with in §3.5..

Example 6

Consider the ODE

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0. \quad \text{Let} \quad \begin{cases} y_1 = e^{2t} \\ y_2 = te^{2t} \end{cases}$$

(1) Check, y_1, y_2 are solutions of this ODE. (2) Prove they form a fundamental set of solutions.

Solution:

$$\begin{cases} \frac{dy_1}{dt} = 2e^{2t}, & \frac{d^2y_1}{dt^2} = 4e^{2t} \\ \frac{dy_2}{dt} = e^{2t}(1+2t) & \frac{d^2y_2}{dt^2} = 4e^{2t}(1+t) \end{cases}$$

So,

$$\begin{cases} \frac{d^2y_1}{dt^2} + 4\frac{dy_1}{dt} + 4y_1 = 4e^{-2t} - 8e^{-2t} + 4e^{-2t} = 0 \\ \frac{d^2y_2}{dt^2} + 4\frac{dy_2}{dt} + 4y_2 = 4e^{2t}(1+t) - 4e^{2t}(1+2t) + 4te^{2t} = 0 \end{cases}$$

So, both y_1, y_2 are solutions.

Continued:

So, the Wronskian: $W(t) =$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}(1+2t) \end{vmatrix} = e^{4t} \neq 0$$

By Theorem 3.3.3 y_1, y_2 form a fundamental set of solutions.

Remark. The CE $r^2 - 4r + 4 = 0$ had a repeated real root $r = 2$, to be dealt with in §3.4.

Abel's Theorem

Consider the 2^{nd}-order homogeneous linear ODE, on an interval $I : \alpha < t < \beta$:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \tag{14}$$

where $p(t), q(t)$ are continuous function on I.

- In the next frame, we state Abel's Theorem, to compute the Wronskian of any two solutions $y = y_1(t)$, $y = y_2(t)$.
- However, it does not seem very useful, because it does not help to determine whether the Wronskian is zero or not?

Theorem 3.3.5

Theorem 3.3.5 Suppose y_1, y_2 are two solutions of (14) and p, q are continuous on the open interval $I : \alpha < t < \beta$. Then,

$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)dt\right) \qquad (15)$$

where c is constant, independent of t, while it depends on y_1, y_2.

Consequently, either $W(y_1, y_2)(t) = 0$ for all t in I (case $c = 0$) or $W(y_1, y_2)(t) \neq 0$ for all t in I.

Proof.

We have

$$\left\{\begin{array}{l} y_1'' + p(t)y_1' + q(t)y_1 = 0 \\ y_2'' + p(t)y_2' + q(t)y_2 = 0 \end{array}\right. \implies$$

$$\left\{\begin{array}{l} y_1''y_2 + p(t)y_1'y_2 + q(t)y_1y_2 = 0 \\ y_2''y_1 + p(t)y_2'y_1 + q(t)y_2y_1 = 0 \end{array}\right. \implies$$

$$(y_2''y_1 - y_1''y_2) = -p(t)(y_2'y_1 - y_1'y_2) = -p(t)W(t) \quad (16)$$

where $W(t) := W(y_1, y_2)(t) = y_2'y_1 - y_1'y_2$.

Continued

$$\text{It turns out} \quad \frac{dW(t)}{dt} = y_2'' y_1 - y_1'' y_2$$

From (16), we get

$$\frac{dW(t)}{dt} = -p(t)W(t) \Longrightarrow \int \frac{dW(t)}{W(t)} = -\int p(t)dt + c_0 \Longrightarrow$$

$$\ln(W(t)) = -\int p(t)dt + c_0 \Longrightarrow W(t) = c\exp\left(-\int p(t)dt\right)$$

The proof is complete.

Example 7

Consider the, general form, 2^{nd}-order Linear Homogeneous ODE, with constant coefficients:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad a \neq 0. \tag{17}$$

Let $y = y_1, y = y_2$ be two solutions. Use Abel's theorem to determine the Wronskian $W(y_1, y_2)$, up to a constant.

Solution:

We rewrite the ODE in the standard form:

$$a\frac{d^2y}{dt^2}+\frac{b}{a}\frac{dy}{dt}+\frac{c}{a}y=0$$

By Abel's Theorem:

$$W(y_1,y_2)=c\exp\left(-\int\frac{b}{a}dt\right)=ce^{-\frac{b}{a}t}$$

- **Remark 1.** The roots of the CE of (17), are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, $-\frac{b}{a}$ =some of the two roots
- **Remark 2.** Unfortunately, we cannot determine c, without further information about y_1, y_2. For this reason, we de-emphasize Abel's Theorem.

Example 8

Consider the 2^{nd}-order Linear Homogeneous ODE

$$(1+t^4)y'' + 4t^3y' + q(t)y = 0$$

Let $y = y_1, y = y_2$ be two solutions. Use Abel's theorem to determine the Wronskian $W(y_1, y_2)$, up to a constant.

- Rewrite the ODE in the standard form

$$y'' + \frac{4t^3}{1+t^4}y' + \frac{q(t)}{1+t^4} = 0$$

Continued

- So, $p(x) = \frac{4t^3}{1+t^4}$.
- By (15), the Wronskian

$$W = c\exp\left(-\int p(t)dt\right) = c\exp\left(-\int \frac{4t^3}{1+t^4}dt\right)$$

$$= c\exp\left(-\ln\left|1+t^4\right|\right) = \frac{c}{1+t^4}$$