Chapter 3: Second Order ODE §3.7 Nonhomogeneous LSODEs Method of Undetermined Coefficients

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Goals

We continue to solve some Nonhomogeneous 2^{nd} order linear ODE, with constant coefficients:

$$\mathcal{L}(y) = ay'' + by' + cy = g(t) \qquad a, b, c \in \mathbb{R}. \qquad (1)$$

We dealt with some problems (in Examples and Homework), by the Method of Variation of Parameters, where $g(t)$ looks like, as described in the next frame!

Goals: Form of $g(t)$

$$g(t) = \begin{cases} e^{\lambda t} \\ \cos \mu t \\ \sin \mu t \\ \text{A Polynomial} \\ \text{A Product of the above.} \end{cases} \tag{2}$$

After solving enough of such problems, with the Method of Variation of Parameters, you see a Pattern evolves, regarding Particular solutions.

Goals: The Pattern of the Particular Solution

For example:

- Whenever $g = ae^{\lambda t}$, we saw the particular solution looked like $Y = Ae^{\lambda t}$. If we believe this, we could substitute $Y = Ae^{\lambda t}$ in the ODE (1), and try our luck in finding A.
- Likewise, when $g(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$, is a polynomial, we may have seen that the particular solution looks like $Y = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$. Again, if we believe this, we could substitute Y in the ODE (1), and try our luck in finding $A_0, A_1, \ldots, A_n$.
- If the fist guess fails, we refine our guess (the pattern).

Goals: The Chart of such Patterns

- Textbooks and Internet are full of such charts for appropriate guess for Y, for a form of $g(t)$, as in (2).
- The students can net search "Method of Undetermined Coefficients" for such a Chart.
- *I would add one more theorem in the next frame, which helps to deal with a wider variety of* $g(t)$, *namely the sum of those given in (2).*

Theorem for $g(t) = g_1(t) + g_2(t)$

Theorem 3.7.1 Let $P(t), Q(t), R(t), g_1(t), g_2(t)$ be function on an interval I. Consider the following three ODE:

$$\begin{cases} P(t)y" + Q(t)y' + R(t)y = g_1(t) \\ P(t)y" + Q(t)y' + R(t)y = g_2(t) \\ P(t)y" + Q(t)y' + R(t)y = g_1(t) + g_2(t) \end{cases}$$

Suppose
$y = Y_1(t)$ is a solution of the first ODE, and
$y = Y_2(t)$ is a solution of the second ODE. Then,
$y = Y_1(t) + Y_2(t)$ is a solution of the third ODE.

Example 1

Give a Particular Solution of the ODE

$$4y'' - 20y' + 25y = 1 + t + t^2 \tag{3}$$

Also give a general solution.

Solution Here $g(t) = 1 + t + t^2$ is a polynomial of degree two. Our first guess is: $Y = A + Bt + Ct^2$. Its derivatives:

$$\begin{cases} Y'(t) = B + 2Ct \\ Y''(t) = 2C \end{cases}$$

Continued

Substituting in (3):

$$4(2C) - 20(B + 2Ct) + 25(A + Bt + Ct^2) = 1 + t + t^2$$

Equating coefficient of t^0, t, t^2, we have

$$\begin{cases} 25C = 1 \\ -40C + 25B = 1 \\ 8C - 20B + 25A = 1 \end{cases} \Longrightarrow \begin{cases} C = \frac{1}{25} \\ B = \frac{13}{125} \\ A = \frac{345}{3125} \end{cases}$$

So, a particular solution is:

$$Y = A + Bt + Ct^2 = \frac{345}{3125} + \frac{13}{125}t + \frac{1}{25}t^2$$

Contingency Plan

Contingency Plan:

- Out First guest $Y = A + Bt + Ct^2$ worked.
- If the first guest did not work, we would try $Y = t(A + Bt + Ct^2)$.
- If that did not work, we would try $Y = t^2(A + Bt + Ct^2)$, and so on!

General Solution

- The CE $4r^2 - 20r + 25 = 0$ has a double root $r = \frac{5}{2}$.
- So, fundamental set of solutions for the homogeneous ODE is

$$\begin{cases} y_1 = e^{\frac{5}{2}t} \\ y_2 = te^{\frac{5}{2}t} \end{cases}$$

- So, a general solution is

$$y = c_1y_1 + c_2y_2 + Y = c_1e^{\frac{5}{2}t} + c_2te^{\frac{5}{2}t} + \left(\frac{345}{3125} + \frac{13}{125}t + \frac{1}{25}t^2\right)$$

Example 2

Give a Particular Solution of the ODE

$$y'' - 4y' + 8y = (1 + t + t^2)e^{2t} \tag{4}$$

Also give a general solution.

Solution Here $g(t)$ is product of e^{2t} and a polynomial $P(t) = 1 + t + t^2$ is a polynomial of degree two. Our first guess is: $Y = e^{2t}(A + Bt + Ct^2)$. Its derivatives:

$$\begin{cases} Y'(t) = & e^{2t}(B + 2Ct) + 2e^{2t}(A + Bt + Ct^2) \\ \quad\quad = & e^{2t}\left((2A + B) + (2B + 2C)t + 2Ct^2)\right) \\ Y''(t) = & e^{2t}\left((2B + 2C) + 4Ct\right) + 2e^{2t}(\cdots) \\ \quad\quad = & e^{2t}\left((4A + 4B + 2C) + (4B + 8C)t + 4Ct^2\right) \end{cases}$$

Continued

Substituting in the ODE (4), we have

$$\left[e^{2t}\left((4A+4B+2C)+(4B+8C)t+4Ct^2\right)\right]$$

$$-4\left[e^{2t}\left((2A+B)+(2B+2C)t+2Ct^2)\right)\right]$$

$$+8\left[e^{2t}(A+Bt+Ct^2)\right] = (1+t+t^2)e^{2t}$$

$$(4A+2C)+4Bt+4Ct^2 = 1+t+t^2 \Longrightarrow$$

Continued

$$\begin{cases} 4A + 2C = 1 \\ 4B = 1 \\ 4C = 1 \end{cases} \Longrightarrow \begin{cases} A = \frac{1}{8} \\ B = \frac{1}{4} \\ C = \frac{1}{4} \end{cases}$$

So, a particular solution is:

$$Y = e^{2t}(A + Bt + Ct^2) = e^{2t}\left(\frac{1}{8} + \frac{1}{4}t + \frac{1}{4}t^2\right)$$

Contingency Plan

Contingency Plan:

- Out First guest $Y = e^{2t}(A + Bt + Ct^2)$ worked.
- If the first guest did not work, we would try
  $Y = te^{2t}(A + Bt + Ct^2)$.
- If that did not work, we would try
  $Y = t^2e^{2t}(A + Bt + Ct^2)$,
  and so on!

General Solution

- The CE $r^2 - 4r + 8 = 0$ has a double root $r = 2 \pm 2i$.
- So, fundamental set of solutions for the homogeneous ODE is
$$\begin{cases} y_1 = e^{2t}\cos 2t \\ y_2 = e^{2t}\sin 2t \end{cases}$$
- So, a general solution is
$$y = c_1y_1 + c_2y_2 + Y$$
$$= c_1e^{2t}\cos 2t + c_2e^{2t}\sin 2t + e^{2t}\left(\frac{1}{8} + \frac{1}{4}t + \frac{1}{4}t^2\right)$$

Example 3

Consider the ODE

$$y" - 2y' + 10y = 37 \sin 3t \tag{5}$$

Give a general solution.

Solution
Here $g(t) = 37 \sin 3t$. Our first guest is that a particular solution has the form

$$Y = A_0 \cos 3t + B_0 \sin 3t.$$

Continued

Two Derivatives of Y are

$$\begin{cases} Y' = -3A_0 \sin 3t + 3B_0 \cos 3t \\ Y'' = -9A_0 \cos 3t - 9B_0 \sin 3t \end{cases}$$

Substituting in (5)

$$(-9A_0 \cos 3t - 9B_0 \sin 3t) - 2(-3A_0 \sin 3t + 3B_0 \cos 3t)$$

$$+10(A_0 \cos 3t + B_0 \sin 3t) = 37 \sin 3t \implies$$

$$\cos 3t(-9A_0 - 6B_0 + 10A_0) + \sin 3t(-9B_0 + 6A_0 + 10B_0)$$

$$= 37 \sin 3t$$

So, $$\begin{cases} A_0 & -6B_0 & = 0 \\ 6A_0 & +B_0 & = 37 \end{cases} \Longrightarrow \begin{cases} A_0 & = 6 \\ B_0 & = 1 \end{cases} \qquad (6)$$

So, $$Y = A_0 \cos 3t + B_0 \sin 3t = 6 \cos 3t + \sin 3t$$

Continued

The CE $r^2 - 2r + 10 = 0$ has solutions $r = 1 \pm 3i$. So, a fundamental set of solutions:

$$\begin{cases} y_1 = e^t \cos 3t \\ y_2 = e^t \sin 3t \end{cases}$$

So, a general solutions is

$$y = c_1y_1 + c_2y_2 + Y = c_1e^t \cos 3t + c_2e^t \sin 3t + (6\cos 3t + \sin 3t)$$

Contingency Plan

Contingency Plan:

- Out First guest $Y = A_0 \cos 3t + B_0 \sin 3t$ worked.
- If the first guest did not work, we would have to refine our guess. In this case refinement may be little more complex to state. A student can look at one of the charts available in the net.