Chapter 3: Second Order ODE
§3.6 Nonhomogeneous LSODEs
Method of Variation of Parameters

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SODEs

- Recall, second order DE (SODE) has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \tag{1}$$

This is also written as

$$y" = f(t, y, y')$$

Nonhomogeneous LSODE

Now, we consider nonhomogeneous LSODEs.

- A nonhomogeneous 2^{nd}-order linear SODE (LSODE), can

$$\text{be written as : } \mathcal{L}(y) = y" + p(t)y' + q(t)y = g(t) \quad (2)$$

  where $p(t), q(t), g(t)$ are functions of t.
- As clarified latter, to solve (2), it would be necessary to solve the the corresponding homogeneous ODE

$$\mathcal{L}(y) = y" + p(t)y' + q(t)y = 0 \qquad (3)$$

- Another form of LSODE (2) is:

$$\mathcal{L}(y) = P(t)y'' + Q(t)y' + R(t)y = G(t) \qquad (4)$$

and the corresponding homogeneous ODE is:

$$\mathcal{L}(y) = P(t)y'' + Q(t)y' + R(t)y = 0 \qquad (5)$$

where $P(t), Q(t), R(t), G(t)$ are functions of t.

- The Plan: We only (mostly) consider problems, so that the corresponding homogeneous ODE (3 or 5) has constant coefficients. We use §3.2, 3.4, 3.5 to solve the respective homogeneous ODE (3 or 5).

Role of the Homogeneous Part

The role of the corresponding homogeneous equation:

- Theorem 3.6.1 Suppose Y_1, Y_2 are two solutions of the nonhomogeneous LSODE (2) or (4):

$$\mathcal{L}(y) = g(t) \qquad or \qquad \mathcal{L}(y) = G(t)$$

  Then, $Y_1 - Y_2$ is a solution of the homogeneous ODE $\mathcal{L}(y) = 0$ (3 or 5).

- Proof. $\mathcal{L}(Y_1) = g(t), \quad \mathcal{L}(Y_2) = g(t) \implies$

$$\mathcal{L}(Y_1 - Y_2) = \mathcal{L}(Y_1) - \mathcal{L}(Y_2) = g(t) - g(t) = 0.$$

The General Solution

Theorem 3.6.2 Suppose Y is a solution of the equation nonhomogeneous LSODE (2) $\mathcal{L}(y) = g(t)$ [*likewise (4)*]. Let y_1, y_2 be a fundamental set of solutions of the homogeneous equation (3) $\mathcal{L}(y) = 0$.
Then, the general solution of (2) [*likewise of* (4)] is:

$$y = \varphi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \qquad (6)$$

where c_1, c_2 are arbitrary constants. Use the notation $y_c = c_1 y_1(t) + c_2 y_2(t)$.

Method of Solutions

Now, we solve some 2^{nd}-order nonhomogeneous ODEs (2, 4). We would only (mostly) consider ODE, with homogeneous part $\mathcal{L}(y) = 0$ with constant coefficients.

By Theorem 3.6.2, two steps would be involved:

- Use methods in §3.2, 3.4, 3.5 to compute a Fundamental set of solutions y_1, y_2 of the homogeneous part $\mathcal{L}(y) = 0$.
- Find a particular solution $Y(t)$ of (2, 4). In this section, we discuss the method of Variation of Parameters, which is discussed next.

Theorem 3.6.3: Variation of Parameters

Theorem 3.6.1: Consider the nonhomogeneous LSODE (2):

$$\mathcal{L}(y) = y'' + p(t)y' + q(t)y = g(t)$$

Assume $p(t), q(t), g(t)$ are continuous on an open interval I. Let y_1, y_2 be a pair of fundamental solutions of the corresponding homogeneous ODE $\mathcal{L}(y) = 0$.

Continued

Then: A particular solution of (2) is given by

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(y_1,y_2)(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(y_1,y_2)(t)} \qquad (7)$$

where these two integrals denote any antiderivatives.
In particular, for numerical solutions, we can take

$$Y = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)ds}{W(y_1,y_2)(s)} + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)ds}{W(y_1,y_2)(s)} \qquad (8)$$

where t_0 is any convenient point in I (Sometimes $t_0 = 0$).

Continued

- So, by (6), the general solution of (2) is

$$y = y_c + Y = (c_1y_1 + c_2y_2) + Y \tag{9}$$

Example 1

Find a general solution of the ODE

$$y'' + 8y' + 16y = 3e^{-t} \tag{10}$$

Step I: Compute Fundamental Set y_1, y_2

- The corresponding homogeneous DE: $y'' + 8y' + 16y = 0$
- The CE: $r^2 + 8r + 16 = 0$.
- So, the CE has a double root $r = -4$.
- From §3.4 a pair of fundamental solutions are:

$$\begin{cases} y_1 = e^{rt} = e^{-4t} \\ y_2 = ty_1 = te^{-4t} \end{cases}$$

- The Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-4t} & te^{-4t} \\ -4e^{-4t} & e^{-4t} - te^{-4t} \end{vmatrix} = e^{-8t}.$$

Step II: Compute Particular Solution

By (7)

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(y_1,y_2)(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(y_1,y_2)(t)}$$

$$= -e^{-4t}\int \frac{te^{-4t}(3e^{-t})dt}{e^{-8t}} + te^{-4t}\int \frac{e^{-4t}(3e^{-t})dt}{e^{-8t}}$$

$$= -e^{-4t}\int 3te^{3t}dt + te^{-4t}\int 3e^{3t}dt$$

$$= -e^{-4t}\left(te^{3t} - \frac{e^{3t}}{3}\right) + te^{-4t}e^{3t} = \frac{e^{-t}}{3}$$

Continued

- So, by (9), the general is

$$y = y_c + Y = (c_1 y_1 + c_2 y_2) + Y$$

$$= c_1 e^{-4t} + c_2 t e^{-4t} + \frac{e^{-t}}{3}$$

Example 2

Find a general solution of the ODE

$$y'' - 2y' - 3y = 4e^{2t} \qquad (11)$$

Step I: Compute a Fundamental Pair y_1, y_2

- The corresponding homogeneous ODE: $y'' - 2y' - 3y = 0$
- The CE: $r^2 - 2r - 3 = 0$.
- So, $r_1 = -1$, $r_2 = 3$
- From §3.1 a pair of fundamental solutions are:

$$\begin{cases} y_1 = e^{-t} \\ y_2 = e^{3t} \end{cases}$$

- The Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{vmatrix} = 4e^{2t}.$$

Step II: Compute a Particular Solution

By (7), with $g(t) = 4e^{2t}$:

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(y_1,y_2)(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(y_1,y_2)(t)}$$

$$= -e^{-t}\int \frac{e^{3t}(4e^{2t})dt}{4e^{2t}} + e^{3t}\int \frac{e^{-t}(4e^{2t})dt}{4e^{2t}}$$

$$= -e^{-t}\frac{e^{3t}}{3} + e^{3t}(-e^{-t}) = -\frac{4e^{2t}}{3}$$

Continued

- So, by (9), the general is

$$y = y_c + Y = (c_1 y_1 + c_2 y_2) + Y$$

$$= c_1 e^{-t} + c_2 e^{3t} - \frac{4e^{2t}}{3}$$

Example 3

Find a general solution of the ODE

$$y'' - 2y' + 5y = 4\cos 2t \tag{12}$$

Step I: Compute a Fundamental Pair y_1, y_2

- The corresponding homogeneous ODE: $y'' + 2y' + y = 0$
- The CE: $r^2 - 2r + 5 = 0$.
- So, $r_1 = 1 + 2i$, $r_2 = 1 - 2i$
- From §3.4 a pair of fundamental solutions are:

$$\begin{cases} y_1 = e^t \cos 2t \\ y_2 = e^t \sin 2t \end{cases}$$

Continued

- The Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^t \cos 2t & e^t \sin 2t \\ e^t \cos 2t - 2e^t \sin 2t & e^t \sin 2t + 2e^t \cos 2t \end{vmatrix} = 2e^t.$$

Step II: Compute a Particular Solution

By (7), with $g(t) = 4\cos 2t$,

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(y_1,y_2)(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(y_1,y_2)(t)}$$

$$= -e^t\cos 2t \int \frac{e^t \sin 2t[4\cos 2t]dt}{2e^t} + e^t \sin 2t \int \frac{e^t\cos 2t[4\cos 2t]dt}{2e^t}$$

$$= -e^t\cos 2t \int 2\sin 2t\cos 2t\ dt + e^t\sin 2t\int 2\cos^2 2t\ dt$$

$$= -e^t\cos 2t\int \sin 4t\ dt + e^t \sin 2t \int (\cos 4t + 1)\ dt$$

Continued

$$Y = -e^t \cos 2t \frac{-\cos 4t}{4} + e^t \sin 2t \left(\frac{\sin 4t}{4} + t \right)$$

$$= -e^t \left(\frac{\cos 6t}{4} - t \sin 2t \right)$$

- So, by (9), the general is

$$y = y_c + Y = (c_1 y_1 + c_2 y_2) + Y$$

$$= c_1 e^t \cos 2t + c_2 e^t \sin 2t + -e^t \left(\frac{\cos 6t}{4} - t \sin 2t \right)$$