Chapter 5:
System of 1^{st}-Order Linear ODE
§5.3 Linear Systems and Eigen Values

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Systems of Linear Equations

Consider a system of m linear equations, in n (unknown) varibales:

$$\begin{array}{cccccc} a_{11}x_1+ & a_{12}x_2+ & a_{13}x_3+ & \cdots+ & a_{1n}x_n & = b_1 \\ a_{21}x_1+ & a_{22}x_2+ & a_{13}x_3+ & \cdots+ & a_{2n}x_n & = b_2 \\ a_{31}x_1+ & a_{32}x_2+ & a_{33}x_3+ & \cdots+ & a_{3n}x_n & = b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}x_1+ & a_{m2}x_2+ & a_{m3}x_3+ & \cdots+ & a_{mn}x_n & = b_m \end{array} \quad (1)$$

where a_{ij}, b_j are real or complex numbers.

Continued

- Write

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

  Then, $\mathbf{A}$ is called the coefficient matrix of the system (1). We also write $\mathbf{A} = (a_{ij})$.
- In matrix form, the system (1) is written as

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{2}$$

The Homogeneous Equation

- If $\mathbf{b} = \mathbf{0}$, then the system (2) would be called a homogeneous system. So,

$$\mathbf{Ax} = \mathbf{0} \tag{3}$$

is a homogeneous system of linear equation.

- Then, $\mathbf{x} = \mathbf{0}$ is a solution of the homogeneous system (3), to be called the trivial solution.

A system and the homogeneous system

- Suppose $\mathbf{x} = \mathbf{x}^{(0)}$ is a solution of the system (2): $\mathbf{Ax} = \mathbf{b}$.
- Then, any solution of (2): $\mathbf{Ax} = \mathbf{b}$ is of the form

$$\mathbf{x} = \mathbf{x}^{(0)} + \xi \quad (4)$$

where ξ is a solution of the corresponding homogeneous system $\mathbf{Ax} = \mathbf{0}$.

Augmented Matrix

- Corresponding to a system (1), define the augmented matrix

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \quad (5)$$

- In deed, the system (1) and the augmented matrix (5) has the same information/data. The Up-shot: the row operations performed on system (1), can be performed on the augmented matrix (5), in stead.

Solving the system (1)

- There are three possibilities:
  - The system (1), have no solution.
  - The system (1), have a unique solution. For this possibility, we need at least n equations.
  - The system (1), have infinitely many solution.
- To solve system (1), we can use TI-84 (ref, rref). Cosult any TI-84 site for instructions.

$n = m$: System of n equations and n unknown

In this course, we focus on the case when $m = n$.
That means, the number of equations is same as number of unknowns $x_1, \ldots, x_n$. Now on, assume $n = m$

- When $n = m$, then the coefficient matrix $\mathbf{A}$ of (1) is a square matrix of size $n \times n$.
- Recall, a square matrix $\mathbf{A}$ is invertible $\iff |\mathbf{A}| \neq 0$.
- If $|A| \neq 0$, then the unique solution of system (2)

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{is} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{6}$$

Linear Indpendence

- A set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of vectors (in $\mathbb{R}^n$) is said to be linearly dependent over $\mathbb{R}$ if there are scalars $c_1, \ldots, c_k$ in $\mathbb{R}$, not all zero such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$.
- Likewise, a set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of vectors (in $\mathbb{C}^n$) is said to be linearly dependent over $\mathbb{C}$ if there are scalars $c_1, \ldots, c_k$ in $\mathbb{C}$, not all zero such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$.
- A set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of vectors is said to be linearly independent over $\mathbb{R}$ or $\mathbb{C}$, if they are not linearly dependent. That means, if

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0} \implies c_1 = c_2 = \cdots = c_k = 0.$$

Continued

- Given a set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$) of vectors, we can form an $n \times k$ matrix $\mathbf{X} := \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_k \end{pmatrix}$.
- Then, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly independent, if $\mathbf{Xc} = \mathbf{0} \Longrightarrow \mathbf{c} = \mathbf{0}$. In other words, $\mathbf{Xc} = \mathbf{0}$ has no non-trivial solution.
- n such vectors, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$)

  $$\textit{are linearly independent} \iff |\mathbf{X}| \neq 0 \iff X \text{is invertible.}$$

Eigenvalues and Eigenvectors

Suppose $\mathbf{A}$ is a square matrix of size $n \times n$.

- A scalar $\lambda \in \mathbb{C}$ is said to be an Eigenvalue of $\mathbf{A}$, if $|\mathbf{A} - \lambda \mathbf{I}| = 0$.
- The following four conditions are equivalent:
  1. $\lambda \in \mathbb{C}$ is an Eigenvalue of $\mathbf{A}$
  2. $|\mathbf{A} - \lambda \mathbf{I}| = 0$
  3. The system $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$ has nontrivial solutions.
  4. There are non-zero vectors $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$.
- Accordingly, a vector $\mathbf{x} \neq \mathbf{0}$ is said to be an eigenvector, for an eigenvalue λ of $\mathbf{A}$, if $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$.

Continued

- Eigenvalues are also called characteristic roots of $\mathbf{A}$. (*The german word "eigen" means "particular" or "peculier"*.)
- The equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$, is a polynomial equation in λ, of degree n, to be called the characteristic equation of $\mathbf{A}$.
- Counting multiplicity of roots, the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$, has n complex roots (including real roots).

Computing Eigen Values and vectors

Matlab can be used to compute eigenvalues and eigenvectors. Consult instructions in my site. The commands eig(A), [V,D]=eig(A) will be useful. However, Matlab does not work too well in this case. Eventually, we will use TI-84 to handle all these. Although, TI-84 does not have any direct command to do all these.

- Sometimes, there is no choice but to use analytic methods. This will be the case, when we have to deal with complex eigenvalues.
- Main thrust of this section is to compute eigenvalues and eigenvectors.

Example 1

Find the eigenvalues and the corresponding eigenvector of

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 4 & -1 \end{pmatrix} \qquad \text{Use Matlab} \quad eig[V, D]$$

- Analytically: The characteristic equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) + 8 = 0 \Longleftrightarrow \lambda^2 + 7 = 0$$

$$\text{Eigenvalues are} \qquad \lambda = \pm\sqrt{7}i$$

Eigenvectors for $\lambda = \sqrt{7}i$

To compute an eigenvector $\lambda = \sqrt{7}i$, we solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-\sqrt{7}i & -2 \\ 4 & -1-\sqrt{7}i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (1-\sqrt{7}i)x_1 - 2x_2 = 0 \\ 4x_1 - (1+\sqrt{7}i)x_2 = 0 \end{cases} \Longrightarrow \begin{cases} (1-\sqrt{7}i)x_1 - 2x_2 = 0 \\ 0 = 0 \end{cases}$$

Continued

So, $x_2 = \frac{1-\sqrt{7}i}{2} x_1$
Taking $x_1 = 1$, an eigenvector for $\lambda = \sqrt{7}i$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-\sqrt{7}i}{2} \end{pmatrix} \tag{7}$$

Eigenvectors for $\lambda = -\sqrt{7}i$

- An eigenvectors for $\lambda = -\sqrt{7}i$ can be computed, as in the case of its conjugate $\lambda = \sqrt{7}i$.
- Alternately, An eigenvectors for $\lambda = -\sqrt{7}i$ is the conjugate of (7):

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+\sqrt{7}i}{2} \end{pmatrix}$$

Example 2

Find the eigenvalues and the corresponding eigenvector of

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}. \qquad \text{Use Matlab} \quad eig[V, D]$$

- The characteristic equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(5-\lambda) + 3 = 0 \iff \lambda^2 - 6\lambda + 8 = 0$$

$$\text{Eigenvalues are} \qquad \lambda = 2, 4$$

Eigenvectors for $\lambda = 2$

For $\lambda = 2$, solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-2 & 3 \\ -1 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + 3x_2 = 0 \\ -x_1 + 3x_2 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = 3x_2 \\ 0 = 0 \end{cases}$$

Continued

Taking $x_2 = 1$, an eigenvector for $\lambda = 2$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad (8)$$

- Since $\lambda = 2$ has multiplicity one, we expect only one linearly independent eigenvector for $\lambda = 2$.

Eigenvectors for $\lambda = 4$

For $\lambda = 4$, solve $(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-4 & 3 \\ -1 & 5-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{array}{c} -3x_1 + 3x_2 = 0 \\ -x_1 + x_2 = 0 \end{array} \right. \implies \left\{ \begin{array}{c} 0 = 0 \\ -x_1 + x_2 = 0 \end{array} \right.$$

Continued

Taking $x_1 = 1$, an eigenvector for $\lambda = 4$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9}$$

- Since $\lambda = 2$ or $\lambda = 4$ has multiplicity one, we expect only one linearly independent eigenvector for, for each.

Example 3

Let

$$A = \begin{pmatrix} -5 & 0 & 0 \\ -1 & 7 & 0 \\ -1 & 1 & 3 \end{pmatrix}.$$

(a) Find the characteristic equation of A, (b) Find all the eigenvalues of A, (c) Corresponding to each eigenvalue, compute an eigen vector.

Solution

Solution: The characteristic polynomial is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+5 & 0 & 0 \\ 1 & \lambda-7 & 0 \\ 1 & -1 & \lambda-3 \end{vmatrix} = (\lambda+5)(\lambda-7)(\lambda-3).$$

So, the characteristic equation is

$$(\lambda+5)(\lambda-7)(\lambda-3) = 0.$$

Therefore, the eigenvalues are $\lambda = -5, 7, 3.$.

Continued

To find an eigenvector corresponding to $\lambda = -5$, solve $(-5I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -12 & 0 \\ 1 & -1 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving, we get

$$x = t, \qquad y = \frac{1}{12}t \qquad z = \frac{1}{8}x - \frac{1}{8}y = \frac{11}{96}t$$

Continued

So, taking $t = 1$, an eigen vector for $\lambda = -5$ is

$$\mathbf{x} = \begin{pmatrix} 1 \\ \frac{1}{12} \\ \frac{11}{96} \end{pmatrix}$$

Continued

To find an eigenvector corresponding to $\lambda = 7$, we have to solve $(7I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} 12 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving, we get

$$x = 0 \qquad y = t \qquad z = \frac{1}{4}(y - x) = \frac{1}{4}t.$$

Continued

With $t = 1$, $\begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix}$ is an eigenvector of A, for eigenvalue $\lambda = 7$.

Continued

To find an eigenvector corresponding to $\lambda = 3$, wehave to solve $(3I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} 8 & 0 & 0 \\ 1 & -4 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\text{So,} \quad x = 0 \qquad y = \frac{1}{4}x = 0 \qquad z = t.$$

With $t = 1$, an eigenvector, for eigenvalue $\lambda = 3$, is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$