Chapter 5:
System of 1^{st}-Order Linear ODE
§5.5 Homogeneous Systems with Constant Coefficients

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Homogeneous System with constant coefficients

- Consider homogeneous linear systems of n Equations, in n variables:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \quad \mathbf{A} \text{ is an } \mathrm{n} \times \mathrm{n} \text{ matrix with constant entries} \tag{1}$$

- As in §5.4, solutions of (1) would be denoted by

$$\mathbf{y}^{(1)}(t) = \begin{pmatrix} y_{11}(t) \\ y_{21}(t) \\ \cdots \\ y_{n1}(t) \end{pmatrix}, \cdots, \mathbf{y}^{(k)}(t) = \begin{pmatrix} y_{1k}(t) \\ y_{2k}(t) \\ \cdots \\ y_{nk}(t) \end{pmatrix}.$$

Principle of superposition

- Recall from §5.4 the Principle of superposition and the converse: If $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ are solution of (1), then, any constant linear combination

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + \cdots + c_n\mathbf{y}^{(n)} \tag{2}$$

  is also a solution of the same system (1).
- The converse is also true, if Wronskian

$$W(t) := W\left(\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}\right)(t) \neq 0$$

Solutions of (1) and Eigenvectors

- Some of the solutions of (1) are given by

$$\mathbf{y} = \xi e^{rt} \quad \textit{where} \quad \mathbf{A}\xi = r\xi, \tag{3}$$

which means that r is an eigenvalue of $\mathbf{A}$ and ξ is an eigenvector, corresponding to r.

- Proof. For $\mathbf{x} = \xi e^{rt}$, we have

$$\mathbf{y}' = (r\xi)e^{rt} = \mathbf{A}\left(\xi e^{rt}\right) = \mathbf{A}\mathbf{y}$$

n eigenvalues and vectors

- Suppose $\mathbf{A}$ has n eigenvalues $r_1, r_2, \ldots, r_n$. Pick eigenvectors ξ_i, corresponding to each r_i. So, $\mathbf{A}\xi_i = r_i\xi_i$.
- A set of n solutions of (1) is given by

$$\mathbf{x}^{(1)} = \xi_1 e^{r_1 t}, \ldots, \mathbf{x}^{(n)} = \xi_n e^{r_n t} \qquad (4)$$

So, the Wronskian

$$W := W(\mathbf{y}^{(1)}, \ldots \mathbf{y}^{(n)}) = \begin{vmatrix} \mathbf{y}^{(1)} & \cdots & \mathbf{y}^{(n)} \end{vmatrix}$$

$$= e^{(r_1+\cdots+r_n)t} \begin{vmatrix} \xi_1 & \cdots & \xi_n \end{vmatrix}$$

Real and Distinct Eigenvalue

- Assume $r_1, r_2, \ldots, r_n$ are distinct. Then, $\xi_1, \ldots, \xi_n$ are linearly independent. (It needs a proof). Hence the Wronskian $W \neq 0$.
- Now assume $r_1, r_2, \ldots, r_n$ are real. By §5.4, $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ form a fundamental set of solutions of (1). In other words, any solution of (1) has the form

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + \cdots + c_n\mathbf{y}^{(n)} = c_1\xi_1 e^{r_1 t} + \cdots + c_n\xi_n e^{r_n t} \quad (5)$$

where $c_1, \ldots, c_n$ are constants, to be determined by the initial value conditions.

Computational Tools

It is evident form (5), to solve problems in this section, we would have to compute eigen values and eigen vectors.

Matlab command [V, D]=eig(A) could be used to find eigenvalues and eigenvectors. Advantage with this is that Matlab can handle complex numbers (see §7.6). However, after experimenting with it, I concluded that it does not work very well. It uses the floating numbers, which make things misleading.

Use Hand computations and TI-84

- Throughout, the following would be my strategy:
  - Compute eigenvalues analytically.
  - If an eigenvalue is real, use use TI-84 (rref) to solve and compute eigenvectors.
  - If an eigenvalue is complex (in §5.6), compute eigenvectors analytically.

Example 1

Find a general solution of

$$\mathbf{y}' = \begin{pmatrix} -6 & 8 \\ -3 & 4 \end{pmatrix} \mathbf{y}$$

- First, find the eigenvalues:

$$\begin{vmatrix} -6-r & 8 \\ -3 & 4-r \end{vmatrix} = 0 \implies r^2 + 2r = 0 \implies r = 0, -2$$

Eigenvectors for $r = 0$

- Eigenvetors for $r = 0$ is given by (use TI-84 "rref"):

$$\begin{pmatrix} -6 & 8 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 0 \\ 1 & -\frac{4}{3} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Taking $\xi_2 = 1$, $\xi = \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix}$ is an eigenvector for $r = 0$.
- So, a solution corresponding to $r = 0$ is:

$$\mathbf{y}^{(1)} = \xi e^{rt} = \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix}$$

Eigenvectors for $r = -2$

- Eigenvetors for $r = -2$ is given by (use TI-84 "rref"):

$$\begin{pmatrix} -6-r & 8 \\ -3 & 4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -4 & 8 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Taking $\xi_2 = 1$, $\xi = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector for $r = -2$.

Continued

- So, a solution corresponding to $r = -2$ is:

$$\mathbf{y}^{(2)} = \xi e^{rt} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t}$$

- So, the general solution is:

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} \qquad (6)$$

Example 2

Find a general solution of

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \mathbf{y}$$

- First, find the eigenvalues:

$$\begin{vmatrix} 2-r & 1 & 1 \\ 1 & 1-r & 2 \\ 1 & 2 & 1-r \end{vmatrix} = 0$$

Continued

$$(2-r)\begin{vmatrix} 1-r & 2 \\ 2 & 1-r \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 1 & 1-r \end{vmatrix} + \begin{vmatrix} 1 & 1-r \\ 1 & 2 \end{vmatrix} = 0$$

$$-r^3 + 4r^2 + r - 4 = 0 \Longrightarrow r^3 - 4r^2 - r + 4 = 0$$

$$r^2(r-1) - 3r(r-1) - 4(r-1) = 0$$

$$(r-1)(r^2 - 3r - 4) = 0 \Longrightarrow (r-1)(r+1)(r-4) = 0$$

$$r = -1, 1, 4$$

An eigenvector and solution for $r = -1$

- Eigenvetors for $r = -1$ is given by (use TI-84 "rref"):

$$\begin{pmatrix} 2-r & 1 & 1 \\ 1 & 1-r & 2 \\ 1 & 2 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \text{(use rref)}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $$\begin{cases} \xi_1 = 0 \\ \xi_2 + \xi_3 = 0 \\ 0 = 0 \end{cases}$$
- Taking $\xi_3 = 1$, $\xi = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector for $r = -1$.
- So, a solution corresponding to $r = -1$ is:

$$\mathbf{y}^{(1)} = \xi e^{rt} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-t}$$

An eigenvector and solution for $r = 1$

- Eigenvetors for $r = 1$ is given by (use TI-84 "rref"):

$$\begin{pmatrix} 2-r & 1 & 1 \\ 1 & 1-r & 2 \\ 1 & 2 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \text{(use rref)}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $$\begin{cases} \xi_1 + 2\xi_3 = 0 \\ \xi_2 - \xi_3 = 0 \\ 0 = 0 \end{cases}$$
- Taking $\xi_3 = 1$, $\xi = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for $r = -1$.
- So, a solution corresponding to $r = 1$ is:

$$\mathbf{y}^{(2)} = \xi e^{rt} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} e^{t}$$

An eigenvector and solution for $r = 4$

- Eigenvetors for $r = 4$ is given by (use TI-84 "rref"):

$$\begin{pmatrix} 2-r & 1 & 1 \\ 1 & 1-r & 2 \\ 1 & 2 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \text{(use rref)}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $$\begin{cases} \xi_1 - \xi_3 = 0 \\ \xi_2 - \xi_3 = 0 \\ 0 = 0 \end{cases}$$
- Taking $\xi_3 = 1$, $\xi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for $r = -1$.
- So, a solution corresponding to $r = 1$ is:
$$\mathbf{y}^{(3)} = \xi e^{rt} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t}$$

General Solution

- So, the general solution is:

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + c_3\mathbf{y}^{(3)}$$

$$= c_1\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t}$$

Example 3

Solve the initial value problem:

$$\mathbf{y}' = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

- First, find the eigenvalues:

$$\begin{vmatrix} 1-r & 3 \\ -1 & 5-r \end{vmatrix} = 0 \implies r^2 - 6r + 8 \implies r = 2, 4$$

Eigenvectors for $r = 2$

- Eigenvetors for $r = 2$ is given by

$$\begin{pmatrix} 1-r & 3 \\ -1 & 5-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{cases} -\xi_1 + 3\xi_2 = 0 \\ 0 = 0 \end{cases}$$

- Taking $\xi_2 = 1$, $\xi = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector for $r = 2$.

A solution corresponding to $r = 2$

- So, a solution corresponding to $r = 2$ is:

$$\mathbf{y}^{(1)} = \xi e^{rt} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t}$$

Eigenvectors for $r = 4$

- Eigenvetors for $r = 4$ is given by

$$\begin{pmatrix} 1-r & 3 \\ -1 & 5-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{cases} 0 = 0 \\ -\xi_1 + \xi_2 = 0 \end{cases}$$

- Taking $\xi_1 = 1$, $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $r = 4$.

A solution corresponding to $r = 4$

- So, a solution corresponding to $r = 4$ is:

$$\mathbf{y}^{(2)} = \xi e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

General Solution

- So, the general solution is:

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

This can be written as

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The Particular Solution

To find the particular solution, use the initial condition:

$$\mathbf{y}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \Longrightarrow$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Longrightarrow \text{(use rref)}$$

to the augmented matrix:

$$c_1 = -1.5, \quad c_2 = 3.5$$

Continued

- So, the particular solution is:

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

$$= -1.5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + 3.5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

Example 4

Solve the Initial Value Problem

$$\mathbf{y}' = \begin{pmatrix} -6 & 8 \\ -3 & 4 \end{pmatrix} \mathbf{y} \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution: This is an extension of an Example above, and general solution was computed (6):

$$\mathbf{y} = c_1 \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} = \begin{pmatrix} \frac{4}{3} & 2e^{-2t} \\ 1 & e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The Particular Solution

To find the particular solution, use the initial condition:

$$\mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \Longrightarrow$$

$$\begin{pmatrix} \frac{4}{3} & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Longrightarrow \text{(use rref)}$$

to the augmented matrix:

$$c_1 = -4.5, \quad c_2 = 3.5$$

Answer

So, the particular solution is:

$$\mathbf{y} = \begin{pmatrix} \frac{4}{3} & 2e^{-2t} \\ 1 & e^{-2t} \end{pmatrix} \begin{pmatrix} -4.5 \\ 3.5 \end{pmatrix} = \begin{pmatrix} 6 + 7e^{-2t} \\ -4.5 + 3.5e^{-2t} \end{pmatrix}$$