Math 220: Differential Equations
Homework and Problems (Solutions)

Satya Mandal

Spring 2018

Contents

Chapter 1

Introduction

1.1 Direction Fields

1. Draw a Direction Field for the DE

$$y' = y \qquad \text{Here y = y(t) is a function of } t.$$

   Pick a suitable window, to show the behavior at $t = \infty$.

2. Draw a Direction Field for the DE

$$y' = -y \qquad \text{Here y = y(t) is a function of } t.$$

   Pick a suitable window, to show the behavior at $t = \infty$.

3. Draw a Direction Field for the DE

$$y' = y - 2 \qquad \text{Here y = y(t) is a function of } t.$$

   Pick a suitable window, to show the behavior at $t = \infty$.

4. Draw a Direction Field for the DE

$$y' = -y - 2 \qquad \text{Here y = y(t) is a function of } t.$$

   Pick a suitable window, to show the behavior at $t = \infty$.

1.2 Solving Some ODEs

In this section, you can use Solution given in Equation 9 in § 1.2

1. Let $y = y(t)$ be a function of t. Solve the initial value problem

$$y' = y \quad y(0) = 100$$

Solution.

$$\frac{dy}{dx} = y \Longrightarrow \int \frac{dy}{y} = \int dx + c \Longrightarrow \ln|y| = x + c$$

Since $y(0) = 100$, We have $\ln 100 = c$. So,

$$\ln|y| = x + \ln 100 \Longrightarrow \ln\left|\frac{y}{100}\right| = x$$

(Assume $y \geq 0$). So

$$\frac{y}{100} = e^x \Longrightarrow y = 100e^x$$

2. Let $y = y(t)$ be a function of t. Solve the initial value problem

$$y' = -y \quad y(0) = 100$$

Solution.

$$\frac{dy}{dx} = -y \Longrightarrow \int \frac{dy}{y} = -\int dx + c \Longrightarrow \ln|y| = -x + c$$

Since $y(0) = 100$, We have $\ln 100 = c$ or $c = 100$. So,

$$\ln\left|\frac{y}{100}\right| = -x + \ln 100$$

(Assume $y \geq 0$). So,

$$\frac{y}{100} = e^{-x} \Longrightarrow y = 100e^{-x}$$

3. Let $y = y(t)$ be a function of t. Solve the initial value problem

$$y' = y - 2 \quad y(0) = 100$$

Solution.

$$\frac{dy}{dx} = y \Longrightarrow \int \frac{dy}{y-2} = \int dx + c \Longrightarrow \ln|y-2| = x + c$$

Since $y(0) = 100$, We have $\ln 98 = c$. So,

$$\ln|y-2| = x + \ln 98 \Longrightarrow \ln\left|\frac{y-2}{98}\right| = x$$

(Assume $y \geq 2$). So

$$\frac{y-2}{98} = e^x \Longrightarrow y = 2 + 98e^x$$

4. Let $y = y(t)$ be a function of t. Solve the initial value problem

$$y' = -y - 2 \quad y(0) = 100$$

Solution.

$$\frac{dy}{dx} = -y \Longrightarrow \int \frac{dy}{y+2} = -\int dx + c \Longrightarrow \ln|y+2| = -x + c$$

Since $y(0) = 100$, We have $\ln 102 = c$. So,

$$\ln|y+2| = -x + \ln 102 \Longrightarrow \ln\left|\frac{y+2}{102}\right| = x$$

(Assume $y \geq 100$). So,

$$\frac{y+2}{102} = e^{-x} \Longrightarrow y = -2 + 102e^{-x}$$

Chapter 2

First Order ODEs

2.1 First Oder Linear ODEs

1. Consider the initial value problem (IVP):

$$\frac{dy}{dt} + 2y = e^{-2t} \qquad y(0) = y_0$$

   (a) Solve the IVP.

   (b) For the solution $y = y(t)$, find the $\lim_{t\to\infty} y(t)$.

   (c) For which values of y_0, the solution stabilizes at infinity?
   (We say $y(t)$ stabilizes at infinity, if $\lim_{t\to\infty} y(t)$ is finite.)

   (d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

Solution: The integrating factor

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int 2dt\right) = e^{2t}$$

So,

$$\frac{d}{dt}\left(y\mu(t)\right) = \mu(t)e^{-2t} = 1 \quad \Longrightarrow y\mu(t) = \int 1 \cdot dt + c = t + c \Longrightarrow$$

$$y = \frac{1}{\mu(t)} [t + c] = e^{-2t} [t + c], \quad y(0) = y_0 \Longrightarrow c = y_0$$

So, the solution is:

$$y = \frac{1}{\mu(t)} [t + c] = e^{-2t} [t + y_0]$$

We have

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left[e^{-2t} [t + y_0]\right] = 0$$

So, the solution stabilizes, for all values of y_0.

2. Consider the initial value problem (IVP):

$$t\frac{dy}{dt} + y = e^{-t} \qquad t > 0 \quad y(1) = y_1$$

   (a) Solve the IVP.
   (b) For the solution $y = y(t)$, find the $\lim_{t\to\infty} y(t)$.
   (c) For which values of y_1, the solution stabilizes at infinity?
   (d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

Solution: Write the equation in the standard form:

$$\frac{dy}{dt} + \frac{y}{t} = \frac{e^{-t}}{t}$$

The integrating factor

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int \frac{1}{t}dt\right) = e^{\ln t} = t$$

So,

$$\frac{d}{dt}(y\mu(t)) = \mu(t)\frac{e^-t}{t} = \frac{e^-t}{t} \Longrightarrow y\mu(t) = \int e^{-t}dt + c = -e^{-t} + c$$

So,

$$y = \frac{1}{\mu(t)} \left[e^{-t} + c\right] = \frac{1}{t}\left[-e^{-t} + c\right] \quad y(1) = y_1 \Longrightarrow y_1 = \left[-e^{-1} + c\right]$$

So, $c = y_1 + e^{-1}$. Therefore,

$$y = \frac{1}{t}\left[-e^{-t} + y_1 + e^{-1}\right]$$

So,

$$\lim_{t\to\infty} y(t) = 0$$

So, the solution stabilizes, for all values of y_1.

3. Consider the initial value problem (IVP):

$$t\frac{dy}{dt} + y = t^2 \qquad t > 0 \quad y(1) = y_1$$

   (a) Solve the IVP.
   (b) For the solution $y = y(t)$, find the $\lim_{t\to\infty} y(t)$.
   (c) For which values of y_1, the solution stabilizes at infinity?
   (d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

Solution: Write the equation in the standard form:

$$\frac{dy}{dt} + \frac{y}{t} = t$$

The integrating factor

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int \frac{1}{t}dt\right) = e^{\ln t} = t$$

So,

$$\frac{d}{dt}\left(y\mu(t)\right) = \mu(t)t = t^2 \Longrightarrow y\mu(t) = \int t^2 dt + c = \frac{t^3}{3} + c$$

So,

$$y = \frac{1}{\mu(t)}\left[\frac{t^3}{3} + c\right] = \frac{1}{t}\left[\frac{t^3}{3} + c\right] \quad y(1) = y_1 \Longrightarrow y_1 = \left[\frac{1}{3} + c\right]$$

So, $c = y_1 - \frac{1}{3}$. Therefore,

$$y = \frac{1}{\mu(t)}\left[\frac{t^3}{3} + c\right] = \frac{1}{t}\left[\frac{t^3}{3} + y_1 - \frac{1}{3}\right] = \frac{t^2}{3} + \frac{y_1}{t} - \frac{1}{3t}$$

So,

$$\lim_{t\to\infty} y(t) = \infty$$

So, the solution FAILS to stabilize, for all values of y_1.

4. Consider the initial value problem (IVP):

$$\frac{dy}{dx} + \frac{1-2x}{x^2}y = 1, \qquad x > 0 \qquad y(1) = y_1$$

   (a) Solve the IVP.

   (b) For the solution $y = y(x)$, find the $\lim_{t\to\infty} y(x)$.

   (c) For which values of y_1, the solution stabilizes at infinity?

   (d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

Solution:

$$\mu(x) = \exp\left(\int p(x)dt\right) = \exp\left(\int \frac{1-2x}{x^2}dx\right) = \exp\left(-\frac{1}{x} - 2\ln x\right)$$

So,

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)dx + c\right) = \exp\left(\frac{1}{x} + 2\ln x\right)\left(\int \exp\left(-\frac{1}{x} - 2\ln x\right)dx + c\right)$$

$$y = e^{\frac{1}{x}}x^2\left(\int e^{\frac{-1}{x}}\frac{1}{x^2}dx + c\right) = e^{\frac{1}{x}}x^2\left(e^{\frac{-1}{x}} + c\right)$$

Use the initial condition:

$$y_1 = e^1(e^{-1} + c) \Longrightarrow e^{-1} + c = e^{-1}y_1 \Longrightarrow c = e^{-1}(y_1 - 1)$$

So,

$$y = e^{\frac{1}{x}}x^2\left(e^{\frac{-1}{x}} + e^{-1}(y_1 - 1)\right) = x^2\left(1 + \frac{y_1 - 1}{e}e^{\frac{1}{x}}\right)$$

So,

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} x^2 \left(1 + \frac{y_1 - 1}{e} e^{\frac{1}{x}}\right) = \lim_{z\to 0} \frac{1}{z^2} \left(1 + \frac{y_1 - 1}{e} e^{z}\right) = \infty$$

So, for all values of y_1 the solution fails to stabilize.

5. Consider the initial value problem (IVP) (Corrected):

$$\frac{dy}{dt} + y\sec^2 t = \tan t \sec^2 t \qquad y(0) = y_0$$

   (a) Solve the IVP.

Solution: The IF

$$\mu(t) = \exp\left(\int \sec^2 t dt\right) = e^{\tan t}$$

So,

$$y\mu(t) = \int e^{\tan t}\tan t \sec^2 t dt + c = \int e^u u du + c \qquad \text{with } u = \tan t \implies$$

$$y\mu(t) = e^u u - e^u + c = e^{\tan t}\tan t - e^{\tan t} + c \implies$$

$$y = \tan t - 1 + ce^{-\tan t}$$

Now $y(0) = y_0$ implies $y_0 = -1 + c$ or $c = 1 + y_0$. So, final solution

$$y = \tan t - 1 + (1 + y_0)e^{-\tan t}$$

The solution is valid in $-\pi/2 < t < \pi/2$.

6. Consider the initial value problem (IVP):

$$\frac{dy}{dx} + (xe^{-x})y = -4xe^{-x} \qquad y(-1) = \Omega$$

   (a) Solve the IVP.
   (b) For the solution $y = y(x)$, find the $\lim_{t\to\infty} y(x)$.
   (c) For which values of Ω, the solution stabilizes at infinity?
   (d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

Solution: The IF

$$\mu(x) = \exp\left(\int xe^{-x}dx\right) = \exp\left(-\int xde^{-x}\right) = \exp\left(-xe^{-x} - e^{-x}\right) = e^{-e^{-x}(x+1)}$$

So,

$$y\mu(x) = \int -4xe^{-x}\left(e^{-e^{-x}(x+1)}\right)dx + c$$

Substitute $u = e^{-x}(x+1)$. So $\frac{du}{dx} = -e^{-x}(x+1) + e^{-x} = -xe^{-x}$. Therefore,

$$y\mu(x) = 4\int e^{-u}du + c = -4e^{-u} + c = -4e^{-e^{-x}(x+1)} + c$$

So,

$$y = \frac{-4e^{-e^{-x}(x+1)} + c}{\mu(x)} = -4 + ce^{e^{-x}(x+1)}$$

The initial value $y(-1) = \Omega$ implies $\Omega = -4 + c$. So, $c = \Omega + 4$. So, the solution of the IVP is

$$y = \frac{-4e^{-e^{-x}(x+1)} + c}{\mu(x)} = -4 + (\Omega + 4)e^{e^{-x}(x+1)}$$

Now,

$$\lim_{x\longrightarrow\infty} y = -4+(\Omega+4)e^{\lim_{x\longrightarrow\infty} e^{-x}(x+1)} = -4+(\Omega+4)e^0 = -4+(\Omega+4)$$

So, the solution is stable, for all values of Ω.

7. Consider the initial value problem (IVP):

$$(1-e^x)\frac{dy}{dx} + 3e^x y = e^x \qquad y(\ln 2) = 0 \qquad \text{(Assume x > 0)}$$

   (a) Solve the IVP.

Solution: In the standard form the ODE is:

$$\frac{dy}{dx} + \frac{3e^x}{(1-e^x)}y = \frac{e^x}{(1-e^x)} \qquad y(\ln 2) = 0$$

The IF

$$\mu(x) = \exp\left(\int \frac{3e^x}{(1-e^x)}dx\right) = \exp\left(-3\ln|u|\right) \quad u = 1-e^x$$

So,

$$\mu(x) = \exp\left(-3\ln(e^x - 1)\right) = \frac{1}{(e^x - 1)^3}$$

Therefore,

$$y\mu(x) = \int \mu(x)\frac{e^x}{(1-e^x)}dx + c = -\int \frac{e^x}{(1-e^x)^4}dx + c = -\frac{1}{3(1-e^x)^3} + c$$

So,

$$y = \frac{1}{3} + c(e^x - 1)^3$$

The initial value $y(\ln 2) = 0$ implies that $c = -\frac{1}{3}$. So, the solutions of the IVP is

$$y = \frac{1}{3} - \frac{1}{3}(e^x - 1)^3$$

8. Consider the initial value problem (IVP):

$$\frac{dy}{dt} + \frac{3y}{t} = \frac{1}{t^2} \qquad t > 0, \quad y(1) = \Omega$$

(a) Solve the IVP.

(b) For the solution $y = y(x)$, find the $\lim_{t\to\infty} y(x)$.

(c) For which values of Ω, the solution stabilizes at infinity?

(d) Optionally, draw the graph of the solution and as well the direction fields of the ODE. And Compare them!

2.2 Separable ODEs

In this section, you can live your answer in implicit form, when it looks too complex to give an explicit solution.

1. Solve the ODE

$$\frac{dy}{dx} = \frac{1+y^2}{yx^2}$$

2. Solve the IVP

$$\frac{dy}{dx} = \frac{x}{y(1-x^2)}, \qquad y(0) = 4$$

3. Solve the IVP
$$\frac{dy}{dx} = y^2(x+2), \qquad y(0) = 1$$

4. Solve the IVP
$$\frac{dy}{dx} = y^2(2x+3x^2), \qquad y(1) = -1$$

5. Solve the ODE
$$\frac{dy}{dt} + y^2 \sec^2 t = 0$$

Solution: Separating the variables, we have
$$\frac{dy}{y^2} = -\sec^2 t dt \quad \Longrightarrow \int \frac{dy}{y^2} = -\int \sec^2 t dt + c \Longrightarrow$$
$$-\frac{1}{y} = \tan t + c$$

6. Solve the IVP
$$\cos y \frac{dy}{dt} + \sec^2 t = 0, \qquad y(0) = 0$$

7. Solve the ODE
$$\tan y \frac{dy}{dt} = 1$$

8. Consider the IVP:
$$\frac{dy}{dx} = \frac{2x+3}{1+y}, \quad y(0) = y_0$$
   (a) Solve the IVP, including the interval in which the solution is valid.
   (b) For the solution $y = y(x)$, find the $\lim_{t\to\infty} y(x)$.
   (c) For which values of y_0, the solution stabilizes at infinity?

Solution: Separating the variables, we have

$$\int (1+y)dy = \int (2x+3)dx + c \Longrightarrow \frac{(1+y)^2}{2} = x^2 + 3x + c$$

It follows, $c = \frac{(1+y_0)^2}{2}$. So,

$$\frac{(1+y)^2}{2} = x^2 + 3x + \frac{(1+y_0)^2}{2}$$

So,

$$y = \pm\sqrt{2x^2 + 6x + (1+y_0)^2} - 1$$

The solution is valid when $2x^2 + 6x + (1+y_0)^2 \geq 0$.
Clearly, $\lim_{t\to\infty} y(x) = \pm\infty$.
So, the solution does not stabilizes at infinity.

2.3 Miscellaneous ODEs

2.3.1 Homogeneous Equations

1. Solve the Homogeneous ODE:

$$\frac{dy}{dx} = \frac{y^3 + 2x^2 y}{x^3} \qquad \text{Assume} \quad x > 0$$

2. Solve the Homogeneous ODE:

$$\frac{dy}{dx} = \frac{5x - 3y}{3x + 5y} \qquad \text{Assume} \quad x > 0, y > 0$$

3. Solve the Homogeneous ODE:

$$\frac{dy}{dx} = \frac{y^3 + xy^2}{yx^2 - x^3} \qquad \text{Assume} \quad x > 0$$

Solution First divide the numerator and denominator by x^3:

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^3 + \left(\frac{y}{x}\right)^2}{\frac{y}{x} - 1} = \frac{v^3 + v^2}{v - 1} \qquad \text{where} \quad v = \frac{y}{x}$$

Substitute $y = xv$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So, the ODE reduces to

$$v + x\frac{dv}{dx} = \frac{v^3+v^2}{v-1} \Longrightarrow x\frac{dv}{dx} = \frac{v^3+v^2}{v-1} - v = \frac{v^3+v}{v-1} = \frac{v(v^2+1)}{v-1} \Longrightarrow$$

$$\int \frac{v-1}{v(v^2+1)}dv = \int \frac{dx}{x} + c = \ln x + c$$

We use method of partial fractions:

$$\int \frac{v-1}{v(v^2+1)}dv = \int \left(\frac{A}{v} + \frac{Bv+C}{v^2+1}\right) dv = \ln x + c \Longrightarrow$$

$$\int \left(\frac{-1}{v} + \frac{v+1}{v^2+1}\right) dv = \ln x + c \Longrightarrow$$

$$-\ln v + \frac{1}{2}\ln(v^2+1) + \tan^{-1} v = \ln x + c \Longrightarrow$$

$$-\ln\left(\frac{y}{x}\right) + \frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2 + 1\right) + \tan^{-1}\left(\frac{y}{x}\right) = \ln x + c \Longrightarrow$$

2.3.2 Bernoulli's Equation

1. Solve the ODE (Bernoulli Equation):

$$\frac{dy}{dx} + \frac{xy}{1+x^2} = x\sqrt{y}$$

Solution: Divide the ODD by $\sqrt{y}$:

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x\sqrt{y}}{1+x^2} = x$$

Substitute $z = \sqrt{y}$. So, $\frac{dz}{dx} = \frac{1}{2\sqrt{y}}\frac{dy}{dx}$. So, the ODE reduces to

$$\frac{dz}{dx} + \frac{xz}{2(1+x^2)} = \frac{x}{2}$$

Integrating factor:

$$\mu(x) = \exp\left(\int \frac{x}{2(1+x^2)}dx\right) = \exp\left(\frac{1}{4}\ln(1+x^2)\right) = (1+x^2)^{1/4}$$

Multiplying the ODE by $\mu(x)$, we have

$$\frac{d}{dx}\left((1+x^2)^{1/4}z\right) = (1+x^2)^{1/4}\frac{x}{2} \Longrightarrow$$

$$(1+x^2)^{1/4}z = \int (1+x^2)^{1/4}\frac{x}{2}dx + c = \frac{1}{4}\int 2x(1+x^2)^{1/4}dx + c \Longrightarrow$$

$$(1+x^2)^{1/4}z = \frac{1}{4}\frac{4}{5}(1+x^2)^{5/4} + c = \frac{1}{5}(1+x^2)^{5/4} + c$$

So, an implicit solution is

$$\sqrt{y} = z = \frac{1}{5}(1+x^2) + c(1+x^2)^{-1/4}$$

So,

$$y = \left(\frac{1}{5}(1+x^2) + c(1+x^2)^{-1/4}\right)^2$$

2. Solve the ODE (Bernoulli Equation):

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

Solution.

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2} \Longrightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{1}{x} = \frac{1}{x^2}$$

Substitute

$$v = \frac{1}{y} \Longrightarrow \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \Longrightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$$

So, the original equation:

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x^2} \Longrightarrow \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2}$$

IF:

$$\mu(x) = \exp\left(\int -\frac{1}{x}dx\right) = \exp(-\ln x) = \exp\left(\ln\left(\frac{1}{x}\right)\right) = \frac{1}{x}$$

So,

$$v = \frac{1}{\mu(x)}\left(\int g(x)\mu(x)dx + c\right) = x\left(\int \frac{-1}{x^2}\frac{1}{x}dx + c\right) = x\left(\frac{1}{2x^2} + c\right)$$

So,

$$v = \frac{1}{2x^2} + cx \Longrightarrow \frac{1}{y} = \frac{1}{2x} + cx$$

3. Solve the ODE (Bernoulli Equation):

$$\frac{dy}{dx} + \frac{y}{2} = \frac{(x-1)y^3}{2}$$

Answer:

$$y^2(x + ce^x) = 1$$

4. Solve the ODE (Bernoulli Equation):

$$\frac{dy}{dx} + \frac{y}{x} = xy^2$$

Solution.

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Longrightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{1}{x} = x$$

Substitute

$$v = \frac{1}{y} \Longrightarrow \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \Longrightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$$

So,

$$-\frac{dv}{dx} + \frac{1}{x}v = x$$

$$\frac{dv}{dx} - \frac{1}{x}v = -x$$

IF

$$\mu(x) = \exp\left(\int -\frac{1}{x}dx\right) = \exp\left(\int -\ln x\right) = \frac{1}{x}$$

So,

$$v = \frac{1}{\mu(x)}\left(\int g(x)\mu(x)dx + c\right) = x\left(\int -x\frac{1}{x}\ dx + c\right) = x\left(-x + c\right)$$

So,

$$\frac{1}{y} = x\left(-x + c\right)$$

2.4 Examples of ODE Models

No Homework

2.5 Existence and Uniqueness of Solutions

1. Consider the initial value problem (IVP)

$$\begin{cases} (t+1)(t-1)(t-2)\frac{dy}{dt} + e^{t^2}y & = \sin t^2 \\ y(-3) & = 1 \end{cases}$$

   Use Theorem 2.5.1 to determine the interval in which this IVP has unique solution. (Do not try to solve).

2. Consider the initial value problem (IVP)

$$\begin{cases} (t+1)(t-1)(t-2)\frac{dy}{dt} + e^{t^2}y & = \cos t^2 \\ y(.5) & = 1 \end{cases}$$

   Use Theorem 2.5.1 to determine the interval in which this IVP has unique solution. (Do not try to solve).

3. Consider the initial value problem (IVP)

$$\begin{cases} \cos t\frac{dy}{dt} + y & = \pi + t^2 \\ y(-\pi) & = 0 \end{cases}$$

   Use Theorem 2.5.1 to determine the interval in which this IVP has unique solution. (Do not try to solve).

4. Consider the initial value problem (IVP)

$$\begin{cases} \cos t \frac{dy}{dt} + y & = \pi + t^2 \\ y(3\pi) & = 0 \end{cases}$$

Use Theorem 2.5.1 to determine the interval in which this IVP has unique solution. (Do not try to solve).

2.6 Equilibrium Solutions

1. Consider the ODE

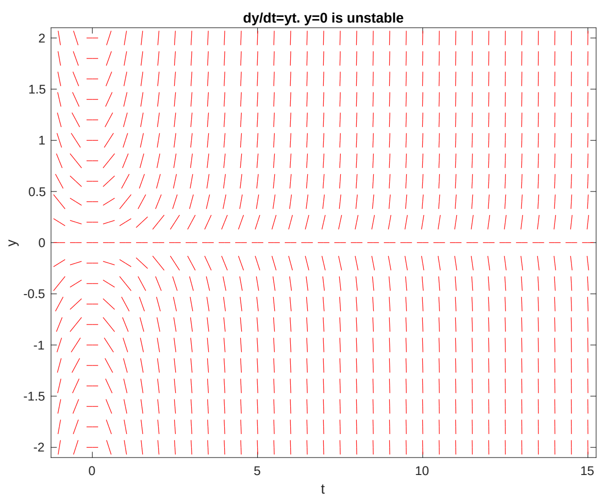
$$\frac{dy}{dt} = yt \qquad\qquad -\infty < y(0) = y_0 < \infty$$

(a) Determine the Equilibrium Solutions.

(b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.

(c) Establish the same analytically, as well.

Solution: You should get used to Matlab. You should also give t the solution, based on sigh. I am going the Matlab solution:



2. Consider the ODE

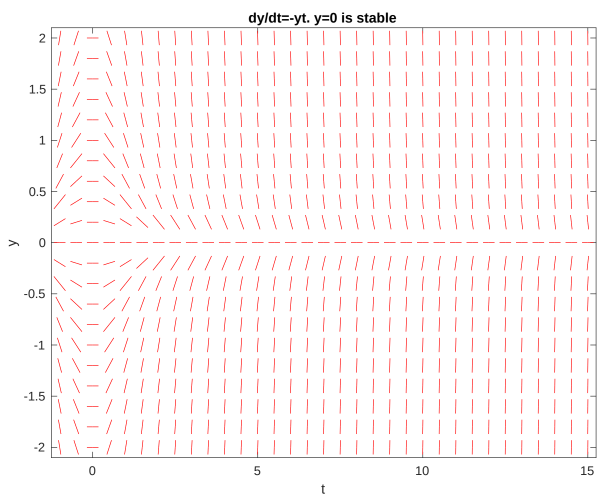
$$\frac{dy}{dt} = -yt \qquad\qquad -\infty < y(0) = y_0 < \infty$$

   (a) Determine the Equilibrium Solutions.

   (b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.

   (c) Establish the same analytically, as well.

   Solution: You should get used to Matlab. You should also give the solution, based on sigh. I am going the Matlab solution:



3. Consider the ODE

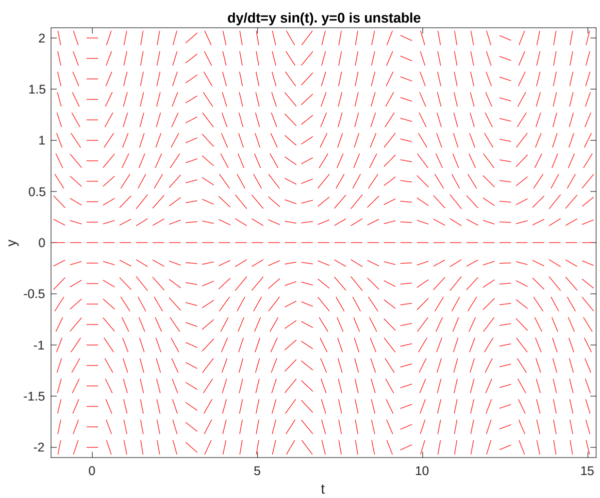
$$\frac{dy}{dt} = y \sin t \qquad\qquad -\infty < y(0) = y_0 < \infty$$

(a) Determine the Equilibrium Solutions.

(b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.

(c) Establish the same analytically, as well.

Solution: You should get used to Matlab. You should also give the solution, based on sigh. I am going the Matlab solution:



4. Consider the ODE

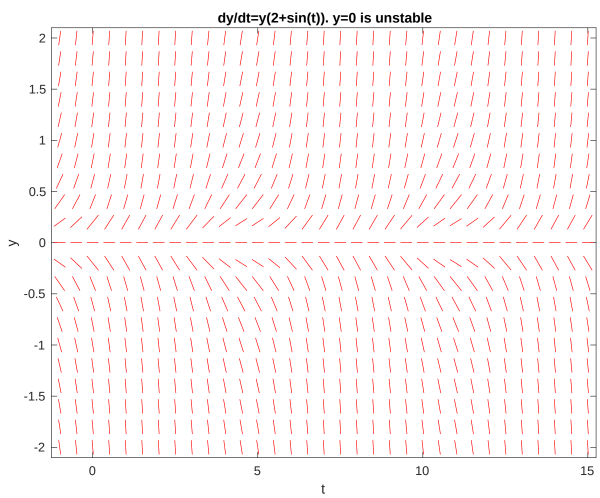
$$\frac{dy}{dt} = y(2 + \sin t) \qquad\qquad -\infty < y(0) = y_0 < \infty$$

   (a) Determine the Equilibrium Solutions.

   (b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.

   (c) Establish the same analytically, as well.

Solution: You should get used to Matlab. You should also give the solution, based on sigh. I am going the Matlab solution:

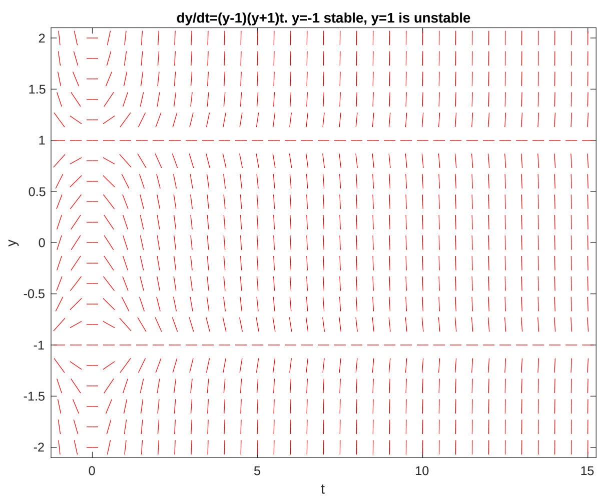


5. Consider the ODE

$$\frac{dy}{dt} = (y+1)(y-1)t \qquad\qquad -\infty < y(0) = y_0 < \infty$$

   (a) Determine the Equilibrium Solutions.
   (b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.
   (c) Optionally, establish the same analytically, as well.

Solution: You should get used to Matlab. You should also give the solution, based on sigh. I am going the Matlab solution:



6. Consider the Autonomous ODE

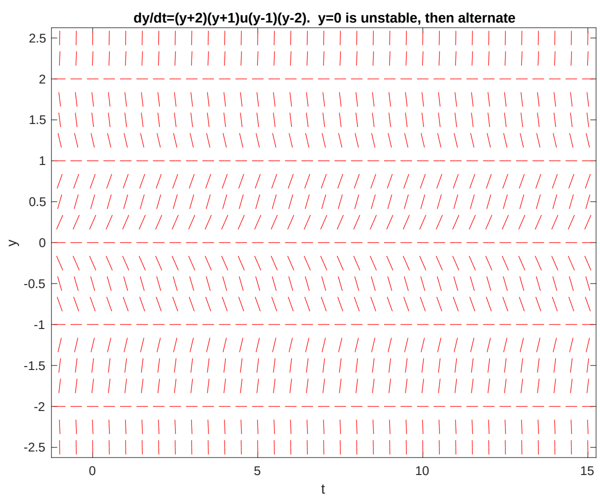
$$\frac{dy}{dt} = (y+2)(y+1)y(y-1)(y-2) \qquad\qquad -\infty < y(0) = y_0 < \infty$$

   (a) Determine the Equilibrium Solutions.

   (b) Classify them as Stable or unstable Equilibrium, using the sign chart and/or Direction Fields.

   (c) Avoid, analytic solution. It may be too time consuming.

Solution: You should get used to Matlab. You should also give the solution, based on sigh. I am going the Matlab solution:



2.7 Exact Equations

1. Prove the following ODEs are not Exact:

   (a) Prove that the following ODE is not exact:

$$\sin(y) + \sin(xy)\frac{dy}{dx} = 0$$

(b) Prove that the following ODE is not exact:

$$\sin(x+y) + \sin(x)\frac{dy}{dx} = 0$$

(c) Prove that the following ODE is not exact:

$$e^{x+y} + xy\frac{dy}{dx} = 0$$

2. Prove that the ODE

$$(x^2 + xy^2 + 4x) + (x^2y - y^2 + y)\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

Solution: Here

$$\begin{cases} M(x,y) = x^2 + xy^2 + 4x \\ N(x,y) = x^2y - y^2 + y \end{cases} \Longrightarrow \begin{cases} \frac{\partial M}{\partial y} = 2xy \\ \frac{\partial N}{\partial x} = 2xy \end{cases} \Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So, the ODE is exact. To solve it, we set

$$\frac{\partial \psi(x,y)}{\partial x} = M = x^2 + xy^2 + 4x, \qquad \frac{\partial \psi(x,y)}{\partial y} = N = x^2y - y^2 + y$$

$$\psi(x,y) = \int M dx + h(y) = \int (x^2 + xy^2 + 4x)dx + h(y) = \frac{x^3}{3} + \frac{x^2y^2}{2} + 2x^2 + h(y)$$

So, $\displaystyle \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3}{3} + \frac{x^2y^2}{2} + 2x^2 + h(y)\right) = x^2y + \frac{dh(y)}{dy} = x^2y - y^2 + y$

So, $\displaystyle \frac{dh(y)}{dy} = -y^2 + y \Longrightarrow h(y) = \int (-y^2 + y)dy = -\frac{y^3}{3} + \frac{y^2}{2}$

Therefore,

$$\psi(x,y) = \frac{x^3}{3} + \frac{x^2y^2}{2} + 2x^2 + h(y) = \frac{x^3}{3} + \frac{x^2y^2}{2} + 2x^2 - \frac{y^3}{3} + \frac{y^2}{2}$$

The general solution of the ODE is:

$$\psi(x,y) = \frac{x^3}{3} + \frac{x^2y^2}{2} + 2x^2 - \frac{y^3}{3} + \frac{y^2}{2} = c$$

3. Prove that the ODE

$$(4x^3 + 3xy^2) + (4y^3 + 3x^2y)\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

4. Prove that the ODE

$$(1 + 6xy^2) + (1 + 6x^2y)\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

Solution: Here

$$\begin{cases} M(x,y) = 1 + 6xy^2 \\ N(x,y) = 1 + 6x^2y \end{cases} \implies \begin{cases} \frac{\partial M}{\partial y} = 12xy \\ \frac{\partial N}{\partial x} = 12xy \end{cases} \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So, the ODE is exact. To solve it, we set

$$\frac{\partial \psi(x,y)}{\partial x} = M = 1 + 6xy^2, \qquad \frac{\partial \psi(x,y)}{\partial y} = N = 1 + 6x^2y$$

$$\text{So,} \quad \psi(x,y) = \int M dx + h(y) = \int (1+6xy^2)dx + h(y) = x + 3x^2y^2 + h(y)$$

$$\text{So,} \quad \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(x + 3x^2y^2 + h(y)\right) = 6x^2y + \frac{dh(y)}{dy} = 1 + 6x^2y$$

$$\text{So,} \quad \frac{dh(y)}{dy} = 1 \implies h(y) = \int dy = y$$

Therefore,

$$\psi(x,y) = x + 3x^2y^2 + h(y) = x + 3x^2y^2 + y$$

The general solution of the ODE is:

$$x + 3x^2y^2 + y = c$$

5. Prove that the ODE

$$\sin(x+y) + (1 + \sin(x+y))\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

Solution: Here

$$\begin{cases} M(x,y) = \sin(x+y) \\ N(x,y) = 1 + \sin(x+y) \end{cases} \Longrightarrow \begin{cases} \frac{\partial M}{\partial y} = \cos(x+y) \\ \frac{\partial N}{\partial x} = 0 + \cos(x+y) \end{cases} \Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So, the ODE is exact. To solve it, we set

$$\frac{\partial \psi(x,y)}{\partial x} = M = \sin(x+y), \qquad \frac{\partial \psi(x,y)}{\partial y} = N = 1 + \sin(x+y)$$

$$\text{So,} \quad \psi(x,y) = \int M dx + h(y) = \int \sin(x+y) dx + h(y) = -\cos(x+y) + h(y)$$

$$\text{So,} \quad \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(-\cos(x+y) + h(y)\right) = \sin(x+y) + \frac{dh(y)}{dy} = 1 + \sin(x+y)$$

$$\text{So,} \quad \frac{dh(y)}{dy} = 1 \Longrightarrow h(y) = \int 1 dy = y$$

Therefore,

$$\psi(x,y) = -\cos(x+y) + h(y) = -\cos(x+y) + y$$

The general solution of the ODE is:

$$\psi(x,y) = -\cos(x+y) + y = c$$

6. Prove that the ODE

$$\cos x \cos y - \sin x \sin y \frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

Solution: Here

$$\begin{cases} M(x,y) = \cos x \cos y \\ N(x,y) = -\sin x \sin y \end{cases} \Longrightarrow \begin{cases} \frac{\partial M}{\partial y} = -\cos x \sin y \\ \frac{\partial N}{\partial x} = -\cos x \sin y \end{cases} \Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So, the ODE is exact. To solve it, we set

$$\frac{\partial \psi(x,y)}{\partial x} = M = \cos x \cos y, \qquad \frac{\partial \psi(x,y)}{\partial y} = N = -\sin x \sin y$$

So, $\psi(x,y) = \int M dx + h(y) = \int \cos x \cos y dx + h(y) = -\sin x \cos y + h(y)$

So, $\dfrac{\partial \psi}{\partial y} = \dfrac{\partial}{\partial y}(-\sin x \cos y + h(y)) = -\sin x \sin y + \dfrac{dh(y)}{dy} = -\sin x \sin y$

$$\text{So,} \quad \frac{dh(y)}{dy} = 0 \Longrightarrow h(y) = \int 0 dy = 0$$

Therefore,

$$\psi(x,y) = -\sin x \cos y + h(y) = -\sin x \cos y$$

The general solution of the ODE is:

$$\psi(x,y) = -\sin x \cos y = c$$

7. Prove that the ODE

$$\left(\ln y + x^2\right) + \left(\frac{x}{y} + 2y\right)\frac{dy}{dx} = 0 \quad \text{is Exact, and solve it.}$$

Solution: Here

$$\begin{cases} M(x,y) = \ln y + x^2 \\ N(x,y) = \frac{x}{y} + 2y \end{cases} \Longrightarrow \begin{cases} \frac{\partial M}{\partial y} = \frac{1}{y} \\ \frac{\partial N}{\partial x} = \frac{1}{y} \end{cases} \Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So, the ODE is exact. To solve it, we set

$$\frac{\partial \psi(x,y)}{\partial x} = M = \ln y + x^2, \qquad \frac{\partial \psi(x,y)}{\partial y} = N = \frac{x}{y} + 2y$$

So, $\psi(x,y) = \int M dx + h(y) = \int \left(\ln y + x^2\right) dx + h(y) = x \ln y + \dfrac{x^3}{3} + h(y)$

$$\text{So,} \quad \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(x \ln y + \frac{x^3}{3} + h(y)\right) = \frac{x}{y} + h(y) = \frac{x}{y} + 2y$$

$$\text{So,} \quad \frac{dh(y)}{dy} = 2y \Longrightarrow h(y) = \int 2y dy = y^2$$

Therefore,

$$\psi(x,y) = x \ln y + \frac{x^3}{3} + h(y) = x \ln y + \frac{x^3}{3} + y^2$$

The general solution of the ODE is:

$$\psi(x,y) = x\ln y + \frac{x^3}{3} + y^2 = c$$

8. Consider the ODE

$$(M_0(x) + M_1(x,y)) + (N_0(y) + N_1(x,y))\frac{dy}{dx} = 0$$

where $M_0(x)$ is a differentiable function of x, $N_0(y)$ is a differentiable function of y, and M_1, N_1 are differentiable functions of x, y. Prove:

$$\text{If} \qquad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}, \qquad \text{then the ODE is Exact.}$$

Solution: Here

$$\begin{cases} M(x,y) = M_0(x) + M_1(x,y) \\ N(x,y) = N_0(y) + N_1(x,y) \end{cases} \quad \text{So,} \quad \begin{cases} \frac{\partial M}{\partial y} = \frac{\partial M_1}{\partial y} \\ \frac{\partial N}{\partial x} = \frac{\partial N_1}{\partial x} \end{cases}$$

So,

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies \text{The ODE is Exact.}$$

2.8 Numerical Solutions: Euler's Method

1. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = 3t^2 \\ y(1) = 1 \end{cases}$$

   (a) Compute the analytic solution $y = \varphi(t)$ and evaluate $\varphi(2)$.

   (b) Use Euler method to approximate $\varphi(2)$, with $h = .05$. Submit the Matlab or Excel output.

2. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = -y + t \\ y(0) = 2 \end{cases}$$

(a) Compute the analytic solution $y = \varphi(t)$ and evaluate $\varphi(1)$.

(b) Use Euler method to approximate $\varphi(1)$, with $h = .05$. Submit the Matlab or Excel output.

Solution: The ODE can be written as $\frac{dy}{dt} + y = t$. With integrating factor $\mu(t) = e^t$, we have

$$e^t y = \int te^t dt + c = te^t - e^t + c \Longrightarrow y = t - 1 + ce^{-t}$$

With $y(0) = 2$, we have

$$y = \varphi(t) = t - 1 + 3e^{-t}. \qquad So, \quad \varphi(1) = 3e^{-1} \approx 1.1036$$

3. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = -y + t \\ y(0) = 1 \end{cases}$$

(a) Compute the analytic solution $y = \varphi(t)$ and evaluate $\varphi(1)$.

(b) Use Euler method to approximate $\varphi(1)$, with $h = .05$. Submit the Matlab or Excel output.

4. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = -y + \sin t \\ y(0) = \frac{1}{2} \end{cases}$$

(a) Compute the analytic solution $y = \varphi(t)$ and evaluate $\varphi(\pi/2)$.

(b) Use Euler method to approximate $\varphi(\pi/2)$, with $h = \frac{\pi}{40}$. Submit the Matlab or Excel output.

Solution: The ODE can be written as $\frac{dy}{dt} + y = \sin t$. With integrating factor $\mu(t) = e^t$, we have

$$e^t y = \int \sin t e^t dt + c = \frac{e^t(\sin t - \cos t)}{2} + c \Longrightarrow$$

$$y = \frac{(\sin t - \cos t)}{2} + ce^{-t}$$

With $y(0) = \frac{1}{2}$, we have

$$y(0) = \frac{1}{2} \Longrightarrow \qquad y = \varphi(t) = \frac{(\sin t - \cos t)}{2} + e^{-t}$$

So, $\varphi(\pi/2) = \frac{1}{2} + \frac{1}{e^{\pi/2}} \approx .7079$

5. Consider the IVP

$$\begin{cases} \frac{dy}{dt} = y^2 + t \\ y(0) = 1 \end{cases}$$

(*We may not have discussed any method to solve this equation analytically.*)

(a) Use Euler method to approximate $\varphi(1)$, with $h = .05$. Submit the Matlab or Excel output.

Chapter 3

Second Order ODE

3.1 Introduction

1. Consider the general form of the Linear Homogenous ODE, of order two:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

   Prove that the constant function $y = \varphi(t) = 0$ is a solution of this equation.

Remark. Note that the above problem is analogous to the following result in Linear Algebra:

Consider the homogeneous system of Linear Equations:

$$A\mathbf{x} = \mathbf{0} \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \quad \text{with } m \text{ rows},$$

and A is a $m \times n$ matrix. Then, $\mathbf{x} = \mathbf{0}$ (with n rows) is a solution of this system.

3.2 Homogeneous Linear second order ODE, with Constant Coefficients

1. Give a general solution of the ODE

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

2. Give a general solution of the ODE

$$9\frac{d^2y}{dx^2} - 9\frac{dy}{dx} - 4y = 0$$

Solution: The CE:

$$9r^2 - 9r - 4 = 0 \Longrightarrow 3r^2 - 12r + 3r - 4 = 0 \Longrightarrow (3r+1)(3r-4) = 0 \Longrightarrow r_1 = -\frac{1}{3},\ r_2 = \frac{4}{3}$$

So,

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{-\frac{t}{3}} + c_2 e^{\frac{4t}{3}}$$

3. Give a general solution of the ODE

$$9\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

Solution: The CE:

$$9r^2 + r = 0 \Longrightarrow r_1 = 0, \quad r_2 = -\frac{1}{9}$$

So,

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 + c_2 e^{-\frac{t}{9}}$$

4. Give a general solution of the ODE

$$4\frac{d^2y}{dx^2} - y = 0$$

Solution: The CE:

$$4r^2 - 1 = 0 \Longrightarrow r_1 = -\frac{1}{2} \quad r_2 = \frac{1}{2}$$

So,

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{-\frac{1}{2}t} + c_2 e^{\frac{1}{2}t}$$

5. Give a general solution of the ODE

$$\frac{d^2y}{dx^2} - \pi \frac{dy}{dt} - 2\pi^2 y = 0$$

Solution: The CE:

$$r^2 - \pi r - 2\pi^2 = 0 \Longrightarrow (r - 2\pi)(r + \pi) = 0 \Longrightarrow \quad r_1 = -\pi, \ r_2 = 2\pi$$

So,

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{-\pi t} + c_2 e^{2\pi t}$$

6. Consider the IVP

$$\begin{cases} \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Solve the IVP $y = \varphi(t)$ and compute $\lim_{t\to\infty} \varphi(t)$.

Solution:

(a) The CE:

$$r^2 + r - 6 = 0 \Longrightarrow (r + 3)(r - 2) = 0 \Longrightarrow r_1 = -3, \ r_2 = 2$$

(b) So, the general solution and its derivative:

$$\begin{cases} y = \varphi(t) = c_1 e^{-3t} + c_2 e^{2t} \\ y' = -3c_1 e^{-3t} + 2c_2 e^{2t} \end{cases}$$

(c) Using the initial values:

$$\begin{cases} c_1 e^0 + c_2 e^0 = 1 \\ -3c_1 e^0 + 2c_2 e^0 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 + c_2 = 1 \\ -3c_1 + 2c_2 = 1 \end{cases} \Longrightarrow \begin{cases} 3c_1 + 3c_2 = 3 \\ -3c_1 + 2c_2 = 1 \end{cases}$$

So, $c_2 = \frac{4}{5}$, $c_1 = \frac{1}{5}$.

(d) So, the solution is

$$y = \varphi(t) = c_1 e^{-3t} + c_2 e^{2t} = \frac{1}{5} e^{-3t} + \frac{4}{5} e^{2t}$$

Also,

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(\frac{1}{5} e^{-3t} + \frac{4}{5} e^{2t} \right) = \infty$$

7. Consider the IVP

$$\begin{cases} \frac{d^2y}{dx^2} - \pi^2 y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Solve the IVP $y = \varphi(t)$ and compute $\lim_{t\to\infty} \varphi(t)$.

Solution:

(a) The CE:

$$r^2 - \pi^2 = 0 \Longrightarrow r_1 = -\pi, \; r_2 = \pi$$

(b) So, the general solution and its derivative:

$$\begin{cases} y = \varphi(t) = c_1 e^{-\pi t} + c_2 e^{\pi t} \\ y' = -\pi c_1 e^{-\pi t} + \pi c_2 e^{\pi t} \end{cases}$$

(c) Using the initial values:

$$\begin{cases} c_1 e^0 + c_2 e^0 = 1 \\ -\pi c_1 e^0 + \pi c_2 e^0 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 + c_2 = 1 \\ -\pi c_1 + \pi c_2 = 1 \end{cases} \Longrightarrow \begin{cases} \pi c_1 + \pi c_2 = \pi \\ -\pi c_1 + \pi c_2 = 1 \end{cases}$$

So,

$$c_2 = \frac{\pi + 1}{2\pi}, \quad c_1 = 1 - \frac{\pi + 1}{2\pi} = \frac{\pi - 1}{2\pi}$$

(d) So, the solution is

$$y = \varphi(t) = c_1 e^{-\pi t} + c_2 e^{\pi t} = \frac{\pi - 1}{2\pi} e^{-\pi t} + \frac{\pi + 1}{2\pi} e^{\pi t}$$

Also,

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(\frac{1}{5} e^{-3t} + \frac{4}{5} e^{2t} \right) = \infty$$

8. Consider the IVP

$$\begin{cases} \frac{d^2y}{dx^2} - 9y = 0 \\ y(0) = \alpha \\ y'(0) = 1 \end{cases}$$

(a) Solve the IVP $y = \varphi(t)$.

(b) For what values of α the limit $\lim_{t\to\infty} \varphi(t)$ is finite?

Solution:

(a) The CE:

$$r^2 - 9 = 0 \Longrightarrow r_1 = -3, \ r_2 = 3$$

(b) So, the general solution and its derivative:

$$\begin{cases} y = \varphi(t) = c_1 e^{-3t} + c_2 e^{3t} \\ y' = -3c_1 e^{-3it} + 3c_2 e^{3t} \end{cases}$$

(c) Using the initial values:

$$\begin{cases} c_1 e^0 + c_2 e^0 = \alpha \\ -3c_1 e^0 + 3c_2 e^0 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 + c_2 = \alpha \\ -3c_1 + 3c_2 = 1 \end{cases} \Longrightarrow \begin{cases} 3c_1 + 3c_2 = 3\alpha \\ -3c_1 + 3c_2 = 1 \end{cases}$$

So,

$$c_2 = \frac{3\alpha + 1}{6}, \quad c_1 = \alpha - \frac{3\alpha + 1}{6} = \frac{3\alpha - 1}{6}$$

(d) So, the solution is

$$y = \varphi(t) = c_1 e^{-3t} + c_2 e^{3t} = \frac{3\alpha - 1}{6} e^{-3t} + \frac{3\alpha + 1}{6} e^{3t}$$

Also,

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(\frac{3\alpha - 1}{6} e^{-3t} + \frac{3\alpha + 1}{6} e^{3t} \right) = \frac{3\alpha - 1}{6} + \frac{3\alpha + 1}{6} \lim_{t\to\infty} e^{3t}$$

THis limit is finite only when $3\alpha + 1$ or $\alpha = -\frac{1}{3}$.

9. Consider the IVP
$$\begin{cases} \frac{d^2y}{dx^2} + 10\frac{dy}{dt} = 0 \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$
Solve the IVP $y = \varphi(t)$ and compute $\lim_{t\to\infty} \varphi(t)$.

Solution:

(a) The CE:
$$r^2 + 10r = 0 \Longrightarrow r_1 = 0,\ r_2 = -10$$

(b) So, the general solution and its derivative:
$$\begin{cases} y = \varphi(t) = c_1 + c_2 e^{-10t} \\ y' = -10c_2 e^{-10t} \end{cases}$$

(c) Using the initial values:
$$\begin{cases} c_1 + c_2 e^0 = 2 \\ -10c_2 e^0 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 + c_2 = 2 \\ c_2 = -\frac{1}{10} \end{cases} \Longrightarrow \begin{cases} c_1 = \frac{21}{10} \\ c_2 = -\frac{1}{10} \end{cases}$$

(d) So, the solution is
$$y = \varphi(t) = c_1 + c_2 e^{-10t} = \frac{21}{10} - \frac{1}{10}e^{-10t}$$

Also,
$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(\frac{21}{10} - \frac{1}{10}e^{-10t}\right) = \frac{21}{10}$$

10. Consider the IVP
$$\begin{cases} \frac{d^2y}{dx^2} - 10\frac{dy}{dt} = 0 \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$
Solve the IVP $y = \varphi(t)$ and compute $\lim_{t\to\infty} \varphi(t)$.

Solution:

(a) The CE:
$$r^2 - 10r = 0 \Longrightarrow r_1 = 0,\ r_2 = 10$$

(b) So, the general solution and its derivative:

$$\begin{cases} y = \varphi(t) = c_1 + c_2 e^{10t} \\ y' = 10c_2 e^{10t} \end{cases}$$

(c) Using the initial values:

$$\begin{cases} c_1 + c_2 e^0 = 2 \\ 10c_2 e^0 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 + c_2 = 2 \\ c_2 = \frac{1}{10} \end{cases} \Longrightarrow \begin{cases} c_1 = \frac{19}{10} \\ c_2 = \frac{1}{10} \end{cases}$$

(d) So, the solution is

$$y = \varphi(t) = c_1 + c_2 e^{10t} = \frac{19}{10} + \frac{1}{10} e^{10t}$$

Also,

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(\frac{19}{10} + \frac{1}{10} e^{10t} \right) = \infty$$

11. Consider the IVP

$$\begin{cases} \frac{d^2y}{dx^2} - 10\frac{dy}{dt} + 21y = 0 \\ y(1) = 0 \\ y'(1) = 0 \end{cases}$$

Solve the IVP.

Solution:

(a) The CE:

$$r^2 - 10r + 21 = 0 \Longrightarrow r_1 = 3, \;\; r_2 = 7$$

(b) So, the general solution and its derivative:

$$\begin{cases} y = \varphi(t) = c_1 e^{3t} + c_2 e^{7t} \\ y' = 3c_1 e^{3t} + 7c_2 e^{7t} \end{cases}$$

(c) Using the initial values:

$$\begin{cases} c_1 e^3 + c_2 e^7 = 0 \\ 3c_1 e^3 + 7e^7 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

(d) So, the solution is

$$y = \varphi(t) = c_1 e^{3t} + c_2 e^{7t} = 0e^{3t} + 0e^{7t} = 0$$

3.3 Fundamental Set of Solutions

1. Consider the 2^{nd}-order linear homogeneous ODE:

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^{-\frac{1}{2}t} \\ y_2 = te^{-\frac{1}{2}t} \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that* y_1, y_2 *are solutions of the ODE.*)

Solution:

(a) The CE: $4r^2 + 4r + 1 = 0$, or $(2r+1)^2 = 0$. It has repeated real root $r = -\frac{1}{2}$. By §3.4, y_1, y_2

(b) The Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{1}{2}t} & te^{-\frac{1}{2}t} \\ -\frac{1}{2}e^{-\frac{1}{2}t} & -\frac{1}{2}te^{-\frac{1}{2}t} + e^{-\frac{1}{2}t} \end{vmatrix}$$

$$= e^{-t} \begin{vmatrix} 1 & t \\ -\frac{1}{2} & -\frac{1}{2}t + 1 \end{vmatrix} = e^{-t}$$

By Wronskian Theorem, y_1, y_2 form a Fundamental set of solutions of the ODE.

2. Consider the 2^{nd}-order linear homogeneous ODE:

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^{-\frac{1}{2}t} \\ y_2 = 7e^{-\frac{1}{2}t} \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that* y_1, y_2 *are solutions of the ODE.*)

Solution: The Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{1}{2}t} & 7e^{-\frac{1}{2}t} \\ -\frac{1}{2}e^{-\frac{1}{2}t} & -7\frac{1}{2}e^{-\frac{1}{2}t} \end{vmatrix}$$

$$= e^{-t} \begin{vmatrix} 1 & 7 \\ -\frac{1}{2} & -\frac{7}{2} \end{vmatrix} = 0$$

By Wronskian Theorem, y_1, y_2 do NOT form a Fundamental set of solutions of the ODE.

3. Consider the 2^{nd}-order linear homogeneous ODE:

$$4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 21y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^{-\frac{3}{2}t} \\ y_2 = e^{\frac{7}{2}t} \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

Solution:

(a) Here the CE: $4r^2 - 8r - 21$. So, $r = \frac{8 \pm \sqrt{64+336}}{8} = \frac{8\pm 20}{8} = \frac{7}{2}, -\frac{3}{2}$. By §3.1, y_1, y_2 are solutions of the ODE.

(b) The Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{3}{2}t} & e^{\frac{7}{2}t} \\ -\frac{3}{2}e^{-\frac{3}{2}t} & \frac{7}{2}e^{\frac{7}{2}t} \end{vmatrix} = e^{2t} \begin{vmatrix} 1 & 1 \\ -\frac{3}{2} & \frac{7}{2} \end{vmatrix} = 5e^{2t} \neq 0$$

By Wronskian Theorem, y_1, y_2 form a Fundamental set of solutions of the ODE.

4. Consider the 2^{nd}-order linear homogeneous ODE:

$$4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 21y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^{\frac{7}{2}t} \\ y_2 = \pi e^{\frac{7}{2}t} \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

Solution:

(a) The Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\frac{7}{2}t} & \pi e^{\frac{7}{2}t} \\ \frac{7}{2}e^{\frac{7}{2}t} & \pi\frac{7}{2}e^{\frac{7}{2}t} \end{vmatrix} = e^{14t}\begin{vmatrix} 1 & \pi \\ \frac{7}{2} & \frac{7}{2}\pi \end{vmatrix} = 0$$

By Wronskian Theorem, y_1, y_2 do NOT form a Fundamental set of solutions of the ODE.

5. Consider the 2^{nd}-order linear homogeneous ODE:

$$\frac{d^2y}{dt^2} + 2y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = \cos(\sqrt{2}t) \\ y_2 = \sin(\sqrt{2}t) \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

Solution:

(a) Here the CE is $r^2 + 2 = 0$, which has complex roots $r = \pm\sqrt{2}i$. So, y_1, y_2 are solutions of this ODE (by §3.5).

(b) The Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\sqrt{2}t) & \sin(\sqrt{2}t) \\ -\sqrt{2}\sin(\sqrt{2}t) & \sqrt{2}\cos(\sqrt{2}t) \end{vmatrix} = \sqrt{2}$$

By Wronskian Theorem, y_1, y_2 form a Fundamental set of solutions of the ODE.

6. Consider the 2^{nd}-order linear homogeneous ODE:

$$\frac{d^2y}{dt^2} + 2y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = \cos(\sqrt{2}t) \\ y_2 = 3\cos(\sqrt{2}t) \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

Solution:

(a) The Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\sqrt{2}t) & 3\cos(\sqrt{2}t) \\ -\sqrt{2}\sin(\sqrt{2}t) & -3\sqrt{2}\sin(\sqrt{2}t) \end{vmatrix} = 0$$

By Wronskian Theorem, y_1, y_2 do NOT form a Fundamental set of solutions of the ODE.

7. Consider the 2^{nd}-order linear homogeneous ODE:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^t\cos(2t) \\ y_2 = e^t\sin(2t) \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

Solution:

(a) Here the CE is $r^2 - 2r + 5 = 0$. By quadratic formula

$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1+2i, 1-2i$$

So, y_1, y_2 are solutions of this ODE (by §3.5).

(b) The Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t\cos(2t) & e^t\sin(2t) \\ e^t\cos(2t) - 2e^t\sin(2t) & e^t\sin(2t) + 2e^t\cos(2t) \end{vmatrix}$$

$$= e^{2t}\begin{vmatrix} \cos(2t) & \sin(2t) \\ \cos(2t) - 2\sin(2t) & \sin(2t) + 2\cos(2t) \end{vmatrix} = 2e^{2t} \neq 0$$

By Wronskian Theorem, y_1, y_2 form a Fundamental set of solutions of the ODE.

8. Consider the 2^{nd}-order linear homogeneous ODE:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0 \quad \text{and its two solutions :} \quad \begin{cases} y_1 = e^t\cos(2t) \\ y_2 = e^t\cos(2t) \end{cases}$$

Use the Wronskian Theorem, to determine if y_1, y_2 form a Fundamental set of solutions of the ODE. (*You need not check that y_1, y_2 are solutions of the ODE.*)

Solution:

(a) Here the CE is $r^2 - 2r + 5 = 0$. By quadratic formula

$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1+2i, 1-2i$$

So, y_1, y_2 are solutions of this ODE (by §3.5).

(b) The Wronskian: $W(y_1, y_2)(t) =$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t\cos(2t) & e^t\cos(2t) \\ e^t\cos(2t) - 2e^t\sin(2t) & e^t\cos(2t) - 2e^t\sin(2t) \end{vmatrix} = 0$$

By Wronskian Theorem, y_1, y_2 do NOT form a Fundamental set of solutions of the ODE.

9. Consider the general form of the 2^{nd}-order linear homogeneous ODE:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad a \neq 0$$

with constant coefficients $a, b, c \in \mathbb{R}$. Let $y = y_1, y = y_2$ be two solution of the ODE. Use Abel's Theorem, to compute the Wronskian $W(y_1, y_2)$, up to a constant multiplier.

Hint: See the same Lemma in §3.3 and reproduce!

3.4 Repeated roots of the CE

1. Solve IVP

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(0) = 3 \\ y'(0) = -1 \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

Solution:

(a) The CE: $4r^2+4r+1=0$. The equation has a double root $r=-\frac{1}{2}$.

(b) So, a fundamental set of solutions is

$$\begin{cases} y_1 = e^{-\frac{1}{2}t} \\ y_2 = te^{-\frac{1}{2}t} \end{cases}$$

(c) The general solution of the IVP and its derivative are

$$\begin{cases} y = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t} \\ y' = -c_1 \frac{1}{2} e^{-\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t} - c_2 \frac{1}{2} t e^{-\frac{1}{2}t} \end{cases}$$

(d) By the initial conditions:

$$\begin{cases} c_1 = 3 \\ -\frac{1}{2}c_1 + c_2 = -1 \end{cases} \implies \begin{cases} c_1 = 3 \\ c_2 = \frac{1}{2} \end{cases}$$

(e) Answer:

$$y = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t} = \left(3 + \frac{1}{2}t\right) e^{-\frac{1}{2}t}$$

Also,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\left(3 + \frac{1}{2}t\right) e^{-\frac{1}{2}t} \right) = 0$$

2. We change the initial condition in the above problem (1a):

Solve IVP

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(1) = 3e^{-\frac{1}{2}} \\ y'(1) = -e^{-\frac{1}{2}} \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

Solution:

(a) From (1a), The general solution of the IVP and its derivative are

$$\begin{cases} y = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t} \\ y' = -c_1 \frac{1}{2} e^{-\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t} - c_2 \frac{1}{2} t e^{-\frac{1}{2}t} \end{cases}$$

(b) Use the initial condition:

$$\left\{\begin{array}{l} c_1e^{-\frac{1}{2}} + c_2e^{-\frac{1}{2}} = 3e^{-\frac{1}{2}} \\ -c_1\frac{1}{2}e^{-\frac{1}{2}} + c_2e^{-\frac{1}{2}} - c_2\frac{1}{2}e^{-\frac{1}{2}t} = -e^{-\frac{1}{2}} \end{array}\right. \implies \left\{\begin{array}{l} c_1 + c_2 = 3 \\ -c_1 + c_2 = -2 \end{array}\right.$$

So, $c_2 = \frac{1}{2}$ and $c_1 = \frac{5}{2}$

(c) Answer:

$$y = c_1e^{-\frac{1}{2}t} + c_2te^{-\frac{1}{2}t} = \left(\frac{5}{2} + \frac{1}{2}t\right)e^{-\frac{1}{2}t}$$

Also,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\left(\frac{5}{2} + \frac{1}{2}t\right)e^{-\frac{1}{2}t}\right) = 0$$

3. Solve IVP

$$25\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + y = 0, \qquad \left\{\begin{array}{l} y(0) = 5 \\ y'(0) = -1 \end{array}\right. \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

Solution:

(a) The CE: $25r^2 - 10r + 1 = 0$. The equation has a double root $r = \frac{1}{5}$.

(b) So, a fundamental set of solutions is

$$\left\{\begin{array}{l} y_1 = e^{\frac{1}{5}t} \\ y_2 = te^{\frac{1}{5}t} \end{array}\right.$$

(c) The general solution of the IVP and its derivative are

$$\left\{\begin{array}{l} y = c_1e^{\frac{1}{5}t} + c_2te^{\frac{1}{5}t} = (c_1 + c_2t)e^{\frac{1}{5}t} \\ y' = c_2e^{\frac{1}{5}t} + \frac{1}{5}(c_1 + c_2t)e^{\frac{1}{5}t} \end{array}\right.$$

(d) By the initial conditions:

$$\left\{\begin{array}{l} c_1 = 5 \\ c_2 + \frac{1}{5}c_1 = -1 \end{array}\right. \implies \left\{\begin{array}{l} c_1 = 5 \\ c_2 = -2 \end{array}\right.$$

(e) Answer:

$$y = (c_1 + c_2 t)e^{\frac{1}{5}t} = (5 - 2t)e^{\frac{1}{5}t}$$

Also,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\left((5 - 2t)e^{\frac{1}{5}t}\right)\right) = -\infty$$

4. We change the initial condition in (5).

Solve IVP

$$25\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(1) = 5e^{\frac{1}{5}} \\ y'(1) = -e^{\frac{1}{5}} \end{cases} \quad \text{Also, compute } \lim_{t\to\infty} y(t).$$

Solution:

(a) From (5) The general solution of the IVP and its derivative are

$$\begin{cases} y = c_1 e^{\frac{1}{5}t} + c_2 t e^{\frac{1}{5}t} = (c_1 + c_2 t)e^{\frac{1}{5}t} \\ y' = c_2 e^{\frac{1}{5}t} + \frac{1}{5}(c_1 + c_2 t)e^{\frac{1}{5}t} \end{cases}$$

(b) Substituting the initial values

$$\begin{cases} (c_1 + c_2)e^{\frac{1}{5}} = 5e^{\frac{1}{5}} \\ c_2 e^{\frac{1}{5}} + \frac{1}{5}(c_1 + c_2)e^{\frac{1}{5}} = -e^{\frac{1}{5}} \end{cases} \Longrightarrow \begin{cases} c_1 + c_2 = 5 \\ c_1 + 6c_2 = -5 \end{cases}$$

So, $c_1 = 7$, $c_2 = -2$

(c) So, answer

$$y = (c_1 + c_2 t)e^{\frac{1}{5}t} = (7 - 2t)\, e^{\frac{1}{5}t}$$

Also,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left((7 - 2t)\, e^{\frac{1}{5}t}\right) = -\infty$$

5. Solve IVP

$$25\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + y = 0, \qquad \begin{cases} y(0) = 5 \\ y'(0) = -1 \end{cases} \quad \text{Also, compute } \lim_{t\to\infty} y(t).$$

Solution:

(a) The CE: $25r^2 + 10r + 1 = 0$. The equation has a double root $r = -\frac{1}{5}$.

(b) So, a fundamental set of solutions is

$$\begin{cases} y_1 = e^{-\frac{1}{5}t} \\ y_2 = te^{-\frac{1}{5}t} \end{cases}$$

(c) The general solution of the IVP and its derivative are

$$\begin{cases} y = c_1 e^{-\frac{1}{5}t} + c_2 te^{-\frac{1}{5}}t = (c_1 + c_2 t)e^{-\frac{1}{5}t} \\ y' = c_2 e^{-\frac{1}{5}t} - \frac{1}{5}(c_1 + c_2 t)e^{-\frac{1}{5}t} \end{cases}$$

(d) By the initial conditions:

$$\begin{cases} c_1 = 5 \\ c_2 - \frac{1}{5}c_1 = -1 \end{cases} \implies \begin{cases} c_1 = 5 \\ c_2 = 0 \end{cases}$$

(e) Answer:

$$y = (c_1 + c_2 t)e^{-\frac{1}{5}t} = 5e^{-\frac{1}{5}t}$$

Also,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\left(5e^{-\frac{1}{5}t}\right)\right) = 0$$

6. Solve IVP

$$\frac{d^2y}{dt^2} + 14\frac{dy}{dt} + 49y = 0, \qquad \begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases} \qquad \text{Also, compute } \lim_{t\to\infty} y(t).$$

Solution:

(a) The CE: $r^2 + 14r + 49 = 0$. The equation has a double root $r = -7$.

(b) So, a fundamental set of solutions is

$$\begin{cases} y_1 = e^{-7t} \\ y_2 = te^{-7t} \end{cases}$$

(c) The general solution of the IVP and its derivative are

$$\begin{cases} y = c_1 e^{-7t} + c_2 te^{-7t} = (c_1 + c_2 t)e^{-7t} \\ y' = c_2 e^{-7t} - 7(c_1 + c_2 t)e^{-7t} \end{cases}$$

(d) By the initial conditions:

$$\begin{cases} c_1 = -1 \\ c_2 - 7c_1 = 1 \end{cases} \implies \begin{cases} c_1 = -1 \\ c_2 = -6 \end{cases}$$

(e) Answer:

$$y = (c_1 + c_2 t)e^{-7t} = (-1 - 6t)e^{-7t}$$

Also,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left((-1 - 6t)e^{-7t}\right) = 0$$

3.5 Complex roots of the CE

1. Solve IVP

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = 0, \qquad \begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases}$$

Also describe the nature of the solution, as $t \to \infty$.

Solution:

(a) The CE: $r^2 - 2\sqrt{5} + 9 = 0$. So, $r = \frac{2\sqrt{5}\pm\sqrt{20-36}}{2} = \sqrt{5} \pm 2i$

(b) So, a Fundamental pair of solutions:

$$\begin{cases} y_1 = e^{\lambda t}\cos \mu t = e^{\sqrt{5}t}\cos 2t \\ y_2 = e^{\lambda t}\sin \mu t = e^{\sqrt{5}t}\sin 2t \end{cases}$$

(c) The general solution of the IVP and its derivative:

$$\begin{cases} y = c_1y_1 + c_2y_2 = c_1e^{\sqrt{5}t}\cos 2t + c_2e^{\sqrt{5}t}\sin 2t = e^{\sqrt{5}t}\left(c_1\cos 2t + c_2\sin 2t\right) \\ y' = \sqrt{5}e^{\sqrt{5}t}\left(c_1\cos 2t + c_2\sin 2t\right) + 2e^{\sqrt{5}t}\left(-c_1\sin 2t + c_2\cos 2t\right) \end{cases}$$

(d) Use the initial value:

$$\begin{cases} c_1 = -1 \\ \sqrt{5}c_1 + 2c_2 = 1 \end{cases} \implies \begin{cases} c_1 = -1 \\ c_2 = \frac{1+\sqrt{5}}{2} \end{cases}$$

(e) So, the answer:

$$y = e^{\sqrt{5}t}\left(c_1\cos 2t + c_2\sin 2t\right) = e^{\sqrt{5}t}\left(-\cos 2t + \frac{1+\sqrt{5}}{2}\sin 2t\right)$$

The exponential part is $E = e^{\sqrt{5}t}$, which goes to ∞. So, it is an unsteady oscillation, at $t \to \infty$.

2. Solve IVP

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = 0, \qquad \begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases}$$

Also describe the nature of the solution, as $t \to \infty$.

Solution:

(a) The CE: $r^2 - 2\sqrt{5}r + 9 = 0$. So, $r = \frac{2\sqrt{5}\pm\sqrt{20-36}}{2} = \sqrt{5} \pm 2i$

(b) So, a Fundamental pair of solutions:

$$\begin{cases} y_1 = e^{\lambda t}\cos \mu t = e^{\sqrt{5}t}\cos 2t \\ y_2 = e^{\lambda t}\sin \mu t = e^{\sqrt{5}t}\sin 2t \end{cases}$$

(c) The general solution of the IVP and its derivative:

$$\begin{cases} y = c_1y_1 + c_2y_2 = c_1e^{\sqrt{5}t}\cos 2t + c_2e^{\sqrt{5}t}\sin 2t = e^{\sqrt{5}t}\left(c_1\cos 2t + c_2\sin 2t\right) \\ y' = \sqrt{5}e^{\sqrt{5}t}\left(c_1\cos 2t + c_2\sin 2t\right) + 2e^{\sqrt{5}t}\left(-c_1\sin 2t + c_2\cos 2t\right) \end{cases}$$

(d) Use the initial value:

$$\begin{cases} c_1 = -1 \\ \sqrt{5}c_1 + 2c_2 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 = -1 \\ c_2 = \frac{1+\sqrt{5}}{2} \end{cases}$$

(e) So, the answer:

$$y = e^{\sqrt{5}t}\left(c_1\cos 2t + c_2\sin 2t\right) = e^{\sqrt{5}t}\left(-\cos 2t + \frac{1+\sqrt{5}}{2}\sin 2t\right)$$

The exponential part is $E = e^{\sqrt{5}t}$, which goes to ∞. So, it is an unstable oscillation, at $t \to \infty$.

3. Solve IVP
$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + (4+\pi^2)y = 0, \qquad \begin{cases} y(1) = -e^2 \\ y'(1) = e^2 \end{cases}$$
Also describe the nature of the solution, as $t \to \infty$.

Solution:

(a) The CE: $r^2 - 4r + (4+\pi^2) = 0$. So, $r = \frac{4 \pm \sqrt{16-4(4+\pi^2)}}{2} = 2 \pm \pi i$

(b) So, a Fundamental pair of solutions:
$$\begin{cases} y_1 = e^{\lambda t}\cos\mu t = e^{2t}\cos\pi t \\ y_2 = e^{\lambda t}\sin\mu t = e^{2t}\sin\pi t \end{cases}$$

(c) The general solution of the IVP and its derivative:
$$\begin{cases} y = c_1y_1 + c_2y_2 = e^{2t}\left(c_1\cos\pi t + c_2\sin\pi t\right) \\ y' = 2e^{2t}\left(c_1\cos\pi t + c_2\sin\pi t\right) + \pi e^{2t}\left(-c_1\sin\pi t + c_2\cos\pi t\right) \end{cases}$$

(d) Use the initial value:
$$\begin{cases} -e^2c_1 = -e^2 \\ -2e^2c_1 - \pi e^2c_2 = e^2 \end{cases} \Longrightarrow \begin{cases} c_1 = 1 \\ c_2 = -\frac{3}{\pi} \end{cases}$$

(e) So, the answer:
$$y = e^{2t}\left(c_1\cos\pi t + c_2\sin\pi t\right) = e^{2t}\left(\cos\pi t - \frac{3}{\pi}\sin\pi t\right)$$

The exponential part is $E = e^{2t}$, which goes to ∞. So, it is an unstable oscillation, at $t \to \infty$.

4. Solve IVP
$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + (4+\pi^2)y = 0, \qquad \begin{cases} y(1) = -e^{-2} \\ y'(1) = e^{-2} \end{cases}$$
Also describe the nature of the solution, as $t \to \infty$.

Solution:

(a) The CE: $r^2+4r+(4+\pi^2)=0$. So, $r=\frac{-4\pm\sqrt{16-4(4+\pi^2)}}{2}=-2\pm\pi i$

(b) So, a Fundamental pair of solutions:

$$\begin{cases} y_1 = e^{\lambda t}\cos\mu t = e^{-2t}\cos\pi t \\ y_2 = e^{\lambda t}\sin\mu t = e^{-2t}\sin\pi t \end{cases}$$

(c) The general solution of the IVP and its derivative:

$$\begin{cases} y = c_1y_1+c_2y_2 = e^{-2t}\left(c_1\cos\pi t + c_2\sin\pi t\right) \\ y' = -2e^{-2t}\left(c_1\cos\pi t + c_2\sin\pi t\right) + \pi e^{-2t}\left(-c_1\sin\pi t + c_2\cos\pi t\right) \end{cases}$$

(d) Use the initial value:

$$\begin{cases} -e^{-2}c_1 = -e^{-2} \\ 2e^{-2}c_1 - \pi e^{-2}c_2 = e^{-2} \end{cases} \implies \begin{cases} c_1 = 1 \\ c_2 = \frac{1}{\pi} \end{cases}$$

(e) So, the answer:

$$y = e^{-2t}\left(c_1\cos\pi t + c_2\sin\pi t\right) = e^{-2t}\left(\cos\pi t + \frac{1}{\pi}\sin\pi t\right)$$

The exponential part is $E=e^{-2t}$, which goes to 0. So, it is an damped oscillation, at $t\to\infty$.

5. Solve IVP

$$\frac{d^2y}{dt^2}+49y=0, \qquad \begin{cases} y(\pi) = -1 \\ y'(\pi) = 7 \end{cases}$$

Also describe the nature of the solution, as $t\to\infty$.

Solution:

(a) The CE: $r^2+49=0$. So, $r=\pm 7i$

(b) So, a Fundamental pair of solutions:

$$\begin{cases} y_1 = e^{\lambda t}\cos\mu t = \cos 7t \\ y_2 = e^{\lambda t}\sin\mu t = \sin 7t \end{cases}$$

(c) The general solution of the IVP and its derivative:

$$\begin{cases} y = c_1y_1+c_2y_2 = c_1\cos 7t + c_2\sin 7t \\ y' = -7c_1\sin 7t + 7c_2\cos 7t \end{cases}$$

(d) Use the initial value:

$$\begin{cases} c_1 \cos 7\pi + c_2 \sin 7\pi = -1 \\ -7c_1 \sin 7\pi + 7c_2 \cos 7\pi = 7 \end{cases} \implies \begin{cases} c_1 = 1 \\ c_2 = -1 \end{cases}$$

(e) So, the answer:

$$y = c_1 \cos 7t + c_2 \sin 7t = \cos 7t - \sin 7t$$

The exponential part is $E = 1$. So, it is an Stable oscillation, at $t \to \infty$.

6. Solve IVP

$$\frac{d^2y}{dt^2} + 4\pi^2 y = 0, \qquad \begin{cases} y(1) = -1 \\ y'(1) = 2 \end{cases}$$

Also describe the nature of the solution, as $t \to \infty$.

Solution:

(a) The CE: $r^2 + 4\pi^2 = 0$. So, $r = \pm 2\pi i$

(b) So, a Fundamental pair of solutions:

$$\begin{cases} y_1 = e^{\lambda t} \cos \mu t = \cos 2\pi t \\ y_2 = e^{\lambda t} \sin \mu t = \sin 2\pi t \end{cases}$$

(c) The general solution of the IVP and its derivative:

$$\begin{cases} y = c_1 y_1 + c_2 y_2 = c_1 \cos 2\pi t + c_2 \sin 2\pi t \\ y' = -2\pi c_1 \sin 2\pi t + 2\pi c_2 \cos 2\pi t \end{cases}$$

(d) Use the initial value:

$$\begin{cases} c_1 \cos 2\pi + c_2 \sin 2\pi = -1 \\ -2\pi c_1 \sin 2\pi + 2\pi c_2 \cos 2\pi = 2 \end{cases} \implies \begin{cases} c_1 = -1 \\ c_2 = \frac{1}{\pi} \end{cases}$$

(e) So, the answer:

$$y = c_1 \cos 2\pi t + c_2 \sin 2\pi t = -\cos 2\pi t + \frac{1}{\pi} \sin 2\pi t$$

The exponential part is $E = 1$. So, it is an Stable oscillation, at $t \to \infty$.

3.6 Nonhomogeneous ODE: Method of Variations of Parameters

For some of the problems, the integration Formula A.1.1, would be helpful.

1. Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = e^{5t}$$

Solution:

(a) The corresponding homogenous ODE: $4y" - 20y' + 25y = 0$.
Its CE: $4r^2 - 20r + 25 = 0$. So, it has a double root, $r = \frac{5}{2}$.
A fundamental set of solutions of the homogeneous ODE:

$$\begin{cases} y_1 = e^{rt} = e^{\frac{5t}{2}} \\ y_2 = te^{rt} = te^{\frac{5t}{2}} \end{cases}$$

The Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\frac{5t}{2}} & te^{\frac{5t}{2}} \\ \frac{5}{2}e^{\frac{5t}{2}} & t\frac{5}{2}e^{\frac{5t}{2}} + e^{\frac{5t}{2}} \end{vmatrix} = e^{5t}$$

(b) A Particular Solutions:

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(t)}$$

$$= -e^{\frac{5t}{2}}\int \frac{te^{\frac{5t}{2}}e^{5t}dt}{e^{5t}} + te^{\frac{5t}{2}}\int \frac{e^{\frac{5t}{2}}e^{5t}dt}{e^{5t}} = -e^{\frac{5t}{2}}\int te^{\frac{5t}{2}}dt + te^{\frac{5t}{2}}\int e^{\frac{5t}{2}}dt$$

$$= -\frac{2}{5}e^{\frac{5t}{2}}\left(te^{\frac{5t}{2}} - \frac{2}{5}e^{\frac{5t}{2}}\right) + \frac{2}{5}te^{\frac{5t}{2}}e^{\frac{5t}{2}} = \frac{4}{25}e^{5t}$$

(c) So, the general solution is

$$y = y_c + Y = c_1y_1 + c_2y_2 + Y = c_1e^{\frac{5t}{2}} + c_2te^{\frac{5t}{2}} + \frac{4}{25}e^{5t}$$

2. Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = e^t$$

Solution:

(a) The corresponding homogeneous ODE $4y? + 4y? + y = 0$
The CE: $4r^2+4r+1 = 0$. The equation has a double root $r = -\frac{1}{2}$.
So, a fundamental set of solutions is

$$\begin{cases} y_1 = e^{-\frac{1}{2}t} \\ y_2 = te^{-\frac{1}{2}t} \end{cases}$$

The Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{1}{2}t} & te^{-\frac{1}{2}t} \\ -\frac{1}{2}e^{-\frac{1}{2}t} & 1t_{\overline{2}}e^{-\frac{1}{2}t} + e^{-\frac{1}{2}t} \end{vmatrix} = e^{-t}$$

(b) A Particular Solutions:

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(t)}$$

$$= -e^{-\frac{1}{2}t}\int \frac{te^{-\frac{1}{2}t}e^t dt}{e^{-t}} + te^{-\frac{1}{2}t}\int \frac{e^{-\frac{1}{2}t}e^t dt}{e^{-t}} = -e^{-\frac{1}{2}t}\int te^{\frac{3}{2}t}dt + te^{-\frac{1}{2}t}\int e^{\frac{3}{2}t}dt$$

$$= -\frac{2}{3}e^{-\frac{1}{2}t}\left(te^{\frac{3}{2}t} - \frac{2}{3}e^{\frac{3}{2}t}\right) + \frac{2}{3}te^{-\frac{1}{2}t}e^{\frac{3}{2}t} = \frac{4}{9}e^t$$

(c) So, the general solution is

$$y = y_c + Y = c_1y_1 + c_2y_2 + Y = c_1e^{-\frac{1}{2}t} + c_2te^{-\frac{1}{2}t} + \frac{4}{9}e^t$$

3. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = e^{10t}$$

Solution:

(a) The corresponding homogenous ODE: $y? + 10y' + 25y = 0$.
Its CE: $r^2 + 10r + 25 = 0$. So, it has a double root, $r = -5$.
A fundamental set of solutions of the homogeneous ODE:

$$\begin{cases} y_1 = e^{rt} = e^{-5t} \\ y_2 = te^{rt} = te^{-5t} \end{cases}$$

The Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-5t} & te^{-5t} \\ -5e^{-5t} & e^{-5t} - 5te^{-5t} \end{vmatrix} = e^{-10t}$$

(b) A Particular Solutions:

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(t)}$$

$$= -e^{-5t}\int \frac{te^{-5t}e^{10t}dt}{e^{-10t}} + te^{-5t}\int \frac{e^{-5t}e^{10t}dt}{e^{-10t}} = -e^{-5t}\int te^{15t}dt + te^{-5t}\int e^{15t}dt$$

$$= -\frac{1}{15}e^{-5t}\left(te^{15t} - \frac{1}{15}e^{15t}\right) + \frac{1}{15}te^{-5t}e^{15t} = \frac{1}{225}e^{15t}$$

(c) So, the general solution is

$$y = y_c + Y = c_1y_1 + c_2y_2 + Y = c_1e^{-5t} + c_2te^{-5t} + \frac{1}{225}e^{15t}$$

4. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = e^{-\sqrt{5}t}$$

Solution:

(a) The CE: $r^2 - 2\sqrt{5} + 9 = 0$. So, $r = \frac{2\sqrt{5}\pm\sqrt{20-36}}{2} = \sqrt{5} \pm 2i$

(b) So, a Fundamental pair of solutions:

$$\begin{cases} y_1 = e^{\lambda t}\cos\mu t = e^{\sqrt{5}t}\cos 2t \\ y_2 = e^{\lambda t}\sin\mu t = e^{\sqrt{5}t}\sin 2t \end{cases}$$

The Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{\sqrt{5}t}\cos 2t & e^{\sqrt{5}t}\sin 2t \\ \sqrt{5}e^{\sqrt{5}t}\cos 2t - 2e^{\sqrt{5}t}\sin 2t & \sqrt{5}e^{\sqrt{5}t}\sin 2t + 2e^{\sqrt{5}t}\cos 2t \end{vmatrix} = 2e^{2\sqrt{5}t}$$

(c) A Particular Solutions:

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(t)}$$

$$= -e^{\sqrt{5}t}\cos 2t \int \frac{e^{\sqrt{5}t}\sin 2tg(t)dt}{2e^{2\sqrt{5}t}} + e^{\sqrt{5}t}\sin 2t \int \frac{e^{\sqrt{5}t}\cos 2tg(t)dt}{2e^{2\sqrt{5}t}}$$

$$= -e^{\sqrt{5}t}\cos 2t \int \frac{e^{-2\sqrt{5}t}\sin 2t}{2}dt + e^{\sqrt{5}t}\sin 2t \int \frac{e^{-2\sqrt{5}t}\cos 2t}{2}dt \qquad \text{(use (A.1.1.))}$$

$$= -e^{\sqrt{5}t}\cos 2t \left(e^{-2\sqrt{5}t}\frac{-2\sqrt{5}\sin 2t - 2\cos \mu t}{48}\right) + e^{\sqrt{5}t}\sin 2t \left(e^{-2\sqrt{5}t}\frac{2\sin 2t - 2\sqrt{5}\cos 2t}{48}\right)$$

$$= \frac{1}{24}e^{-\sqrt{5}t}$$

(d) So, the general solution is

$$y = y_c + Y = c_1y_1 + c_2y_2 + Y = c + Y = c_1e^{\sqrt{5}t}\cos 2t + c_2e^{\sqrt{5}t}\sin 2t + \frac{1}{20}e^{-\sqrt{5}t}$$

5. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + (4+\pi^2)y = 1$$

Solution:

(a) The CE: $r^2 - 4r + (4+\pi^2) = 0$. So, $r = \frac{4\pm\sqrt{16-4(4+\pi^2)}}{2} = 2 \pm \pi i$
So, a Fundamental pair of solutions:

$$\begin{cases} y_1 = e^{\lambda t}\cos \mu t = e^{2t}\cos \pi t \\ y_2 = e^{\lambda t}\sin \mu t = e^{2t}\sin \pi t \end{cases}$$

The Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2t}\cos\pi t & e^{2t}\sin\pi t \\ 2e^{2t}\cos\pi t - \pi e^{2t}\sin\pi t & 2e^{2t}\sin\pi t + \pi e^{2t}\cos\pi t \end{vmatrix} = \pi e^{4t}$$

(b) A Particular Solutions:

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(t)}$$

$$= -e^{2t}\cos\pi t \int \frac{e^{2t}\sin\pi t dt}{\pi e^{4t}} + e^{2t}\sin\pi t \int \frac{e^{2t}\cos\pi t dt}{\pi e^{4t}}$$

$$= -e^{2t}\cos\pi t \int \frac{e^{-2t}\sin\pi t dt}{\pi} + e^{2t}\sin\pi t \int \frac{e^{-2t}\cos\pi t dt}{\pi} \qquad \text{(use (A.1.1.))}$$

$$= -\frac{1}{\pi}e^{2t}\cos\pi t\left(e^{-2t}\frac{-2\sin\pi t - \pi\cos\pi t}{4+\pi^2}\right) + \frac{1}{\pi}e^{2t}\sin\pi t\left(e^{-2t}\frac{\pi\sin\mu t - 2\cos\pi t}{4+\pi^2}\right)$$

$$= \frac{1}{4+\pi^2}e^{2t}$$

(c) So, the general solution is

$$y = y_c + Y = c_1y_1 + c_2y_2 + Y = c + Y = c_1e^{2t}\cos\pi t + c_2e^{2t}\sin\pi t + \frac{1}{4+\pi^2}e^{2t}$$

6. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 49y = 2\sin 7t$$

Solution:

(a) The CE: $r^2 + 49 = 0$. So, $r = \pm 7i$
So, a Fundamental pair of solutions:

$$\begin{cases} y_1 = e^{\lambda t}\cos\mu t = \cos 7t \\ y_2 = e^{\lambda t}\sin\mu t = \sin 7t \end{cases}$$

The Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 7t & \sin 7t \\ -7\sin 7t & 7\cos 7t \end{vmatrix} = 7$$

(b) A Particular Solutions:

$$Y = -y_1(t)\int \frac{y_2(t)g(t)dt}{W(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(t)}$$

$$= -\cos 7t \int \frac{2\sin^2 7tdt}{7} + \sin 7t \int \frac{2\cos 7t \sin 7tdt}{7}$$

$$= -\cos 7t \int \frac{(1-\cos 14t)dt}{7} + \sin 7t \int \frac{\sin 14tdt}{7}$$

$$= -\cos 7t \frac{(t - \frac{\sin 14t}{14})}{7} - \sin 7t \frac{\cos 14t}{98}$$

$$= \frac{-t\cos 7t + (\cos 7t \sin 14t - \sin 7t \cos 14t)}{98} = \frac{-t\cos 7t + \sin 7t}{98}$$

(c) So, the general solution is

$$y = y_c + Y = c_1 y_1 + c_2 y_2 + Y = c + Y = c_1 \cos 7t + c_2 \sin 7t + \frac{-t\cos 7t + \sin 7t}{98}$$

7. Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = \cos 5t$$

Solution:

(a) The corresponding homogenous ODE: $y? - 5y' + 6y = 0$.
Its CE: $r^2 - 5r + 6 = 0$. So, it has a double root, $r = 2, 3$.
A fundamental set of solutions of the homogeneous ODE:

$$\begin{cases} y_1 = e^{rt} = e^{2t} \\ y_2 = te^{rt} = e^{3t} \end{cases}$$

The Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

(b) A Particular Solutions:

$$
\begin{aligned}
Y &= -y_1(t)\int \frac{y_2(t)g(t)dt}{W(t)} + y_2(t)\int \frac{y_1(t)g(t)dt}{W(t)} \\
&= -e^{2t}\int \frac{e^{3t}\cos 5tdt}{e^{5t}} + e^{3t}\int \frac{e^{2t}\cos 5tdt}{e^{5t}} \\
&= -e^{2t}\int e^{-2t}\cos 5tdt + e^{3t}\int e^{-3t}\cos 5tdt \qquad \text{(use (A.1.1.))} \\
&= -e^{2t}\left(e^{-2t}\frac{5\sin 5t - 2\cos 5t}{29}\right) + e^{3t}\left(e^{-3t}\frac{5\sin 5t - 3\cos 5t}{34}\right) \\
&= -\frac{5\sin 5t - 2\cos 5t}{29} + \frac{5\sin 5t - 3\cos 5t}{34}
\end{aligned}
$$

(c) So, the general solution is

$$
\begin{aligned}
y &= y_c + Y = c_1y_1 + c_2y_2 + Y \\
&= c_1e^{2t} + c_2te^{3t} + \frac{4}{25}e^{5t} + \left(-\frac{5\sin 5t - 2\cos 5t}{29} + \frac{5\sin 5t - 3\cos 5t}{34}\right)
\end{aligned}
$$

3.6.1 More Problems

Do not submit the following of the problems in the subsection. The following problems are variations of the above problems, where right hand side $g(t)$ would be different. Majority of the steps would be same as above, while the integrations would be more involved.

1. The following are variations of Problem 1.

   (a) Find the general solution of the ODE

   $$4\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = (1 + t + t^2)e^{5t}$$

   (b) Find the general solution of the ODE

   $$4\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = e^{5t}\cos 3t$$

(c) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = (1+t+t^2)\cos 3t$$

2. The following are variations of Problem 2.

(a) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = (1+t-t^2)e^t$$

(b) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = e^t \sin 2t$$

(c) Find the general solution of the ODE

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = (1+t+t^2)\sin 2t$$

3. The following are variations of Problem 3.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = (1+t-t^2)e^{10t}$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = e^{10t}\cos 3t$$

(c) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = (1+t+t^2)\cos 3t$$

4. The following are variations of Problem 4.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = (1+t+t^2)e^{-\sqrt{5}t}$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = e^{-\sqrt{5}t}\sin 2t$$

(c) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 2\sqrt{5}\frac{dy}{dt} + 9y = (1+t+t^2)\sin 2t$$

5. The following are variations of Problem 5.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + (4+\pi^2)y = 1+t+t^2$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + (4+\pi^2)y = (1+t+t^2)\sin 2t$$

6. The following are variations of Problem 6.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 49y = 2t^2\sin 7t$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} + 49y = 2e^{3t}\sin 7t$$

7. The following are variations of Problem 7.

(a) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = (1+t+t^2)\cos 5t$$

(b) Find the general solution of the ODE

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = e^{t}\cos 5t$$

3.7 Method of Undetermined Coefficients

No new problems will be assigned in the section. One can try to solve any of the problems in Section 3.6 or Section 3.6.1, using this method of undetermined coefficients.

3.8 Elements of Particle Dynamics

We would not assign any Homework in this section. Problems are essentially covered by what we did in § 3.5, 3.6, 3.6.1, 3.7.

The reason for this departure from the customary practice is two fold. The problem sets on this topic in the literature appears a little artificial, to me. Some of problems are, essentially same as those in § 3.5, 3.6, 3.6.1, 3.7, encased within a story on Mechanics. Other set of problems, ask to compute Amplitude, Periodicity etc., which may belong in the Mechanics classes.

Chapter 4

Higher Order ODE

4.1 General Overview of Theory

No Homework

4.2 Linear Homogeneous ODE with constant coefficients

1. Give a general solution of the Homogeneous ODE

$$\frac{d^3y}{dt^3} - y = 0$$

Solution. The CE

$$\begin{cases} r^3 - 1 = 0 \\ (r-1)(r^2+r+1) = 0 \\ r_1 = 1,\ r_2 = e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i\ r_3 = e^{\frac{4\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{cases}$$

r_2 and r_3 are conjugates. So, a fundamental set of solutions

$$y_1 = e^t, \qquad \begin{cases} y_2 = e^{-\frac{1}{2}t}\cos\left(\frac{2\pi}{3}t\right) \\ y_3 = e^{-\frac{1}{2}t}\sin\left(\frac{2\pi}{3}t\right) \end{cases}$$

So, a general solution

$$y = c_1 e^t + c_2^{-\frac{1}{2}t} \cos\left(\frac{2\pi}{3}t\right) + c_3^{-\frac{1}{2}t} \sin\left(\frac{2\pi}{3}t\right)$$

2. Give a general solution of the Homogeneous ODE

$$\frac{d^6 y}{dt^6} - y = 0$$

Solution. The CE

$$\begin{cases} r^6 - 1 = 0 \\ r_1 = 1,\ r_2 = e^{\frac{2\pi}{6}i}, r_3 = e^{2\frac{2\pi}{6}i}, r_4 = e^{3\frac{2\pi}{6}i}, r_5 = e^{4\frac{2\pi}{6}i}, r_6 = e^{5\frac{2\pi}{6}i} \\ r_1 = 1,\ r_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, r_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, r_4 = -1,\ r_5 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, r_6 = \frac{1}{2} - \frac{\sqrt{3}}{2}i, \end{cases}$$

Regrouping the roots with their conjugates:

$$\begin{cases} r_1 = 1 \\ r_4 = -1 \end{cases} \quad \begin{cases} r_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ r_6 = \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{cases} \quad \begin{cases} r_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ r_5 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \end{cases}$$

So, a general solution

$$y = c_1 e^t + c_2^{-\frac{1}{2}t} \cos\left(\frac{2\pi}{3}t\right) + c_3^{-\frac{1}{2}t} \sin\left(\frac{2\pi}{3}t\right)$$

So, a fundamental set of solutions

$$\begin{cases} y_1 = e^t \\ y_2 = e^{-t} \end{cases} \quad \begin{cases} y_3 = e^{\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ y_4 = e^{\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) \end{cases} \quad \begin{cases} y_5 = e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ y_6 = e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) \end{cases}$$

So, the general solution

$$y = c_1 e^t + c_2 e^{-t} + c_3 \left(e^{\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$+c_3 \left(e^{\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)\right) + c_5 \left(e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right)\right) + c_6 \left(e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

3. Give a general solution of the Homogeneous ODE

$$\frac{d^4y}{dt^4} - \pi^4 y = 0$$

4. Give a general solution of the Homogeneous ODE

$$\frac{d^4y}{dt^4} - 4\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 16\frac{dy}{dt} - 32y = 0$$

Solution. The CE

$$\begin{cases} r^4 - 4r^3 + 4r^2 + 16r - 32 = 0 \\ r^3(r-2) - 2r^2(r-2) + 16(r-2) = 0 \\ (r-2)(r^3 - 2r^2 + 16) = 0 \\ (r-2)[r^3 + 2r^2 - 4r^2 - 8r + 8r + 16] = 0 \\ (r-2)(r+2)(r^2 - 4r + 8] = 0 \end{cases}$$

So,

$$r_1 = -2, r_2 = 2, \ r_3, r_4 = \frac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2i$$

So, a fundamental set of solutions

$$y_1 = e^{-2t}, \quad y_2 = e^{2t}, \quad \begin{cases} y_3 = e^{2t}\cos 2t \\ y_4 = e^{2t}\sin 2t \end{cases}$$

So, a general solution

$$y = c_1 e^{-2t} + c_2 e^{2t} + c_3 e^{2t}\cos 2t + c_4 e^{2t}\sin 2t$$

4.3 Nonhomogeneous Linear ODE

No Homework

Chapter 5

System of 1^{st}-Order Linear ODE

5.1 Introduction

No Homework

5.2 Algebra Of Matrices

No Homework

5.3 Linear Systems and Eigen Values

Definition 5.3.1. Let A be a square matrix of order n, with real entries, and $\lambda \in \mathbb{R}$ be a Eigen value of A. Then, the Eigen Space $E(\lambda)$ is defined to be the set of all Eigen Vectors corresponding to λ, together with the zero vector. So,

$$E(\lambda) = \{\mathbf{x} \in \mathbb{R}^n : (A - \lambda I_n)\mathbf{x} = \mathbf{0}\}$$

Note, $E(\lambda)$ is a subspace of $\mathbb{R}^n$.

If $\lambda \in \mathbb{C}$ is a complex Eigen value of A, then the Eigen Space $E(\lambda)$ is defined to be

$$E(\lambda) = \{\mathbf{x} \in \mathbb{C}^n : (A - \lambda I_n)\mathbf{x} = \mathbf{0}\}$$

1. Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

   (a) Write down the characteristic equation of A

   (b) Find all the eigen values of A.

   (c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

2. Let

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

   (a) Write down the characteristic equation of A

   (b) Find all the eigen values of A.

   (c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

3. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

   (a) Write down the characteristic equation of A

   (b) Find all the eigen values of A.

   (c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

4. Let

$$A = \begin{pmatrix} 1 & 2 & -6 \\ -2 & 5 & -6 \\ -2 & 2 & -3 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

5. Let

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 4 & 1 & -2 \\ -4 & 2 & 5 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

6. Let

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

7. Let

$$A = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

8. Let

$$A = \begin{pmatrix} 4 & 3 & -5 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigen values of A.

(c) For each eigen value λ, compute the eigen space $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

5.4 The Theoretical Foundation

No Homework

5.5 Homogeneous Systems with Constant Coefficients

Consider homogeneous systems $\mathbf{y}' = A\mathbf{y}$, where A a constant matrix, of size $n \times n$. The section deals with problems, such that the roots of the characteristic Equation $|A - \lambda I| = 0$ are real and distinct. Consequently, the corresponding eigen vectors would be linearly independent, which lead to a Fundamental Set of Solutions.

1. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 6 & 3 \\ 10 & 5 \end{pmatrix} \mathbf{y}$$

Solution First, the eigen values:

$$\begin{vmatrix} 6-r & 3 \\ 10 & 5-r \end{vmatrix} = 0 \quad \Longrightarrow \quad r^2 - 11r = 0 \Longrightarrow \quad r = 0, \; r = 11$$

(a) An eigenvector foe $r = 0$ is given by $\mathbf{A}\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 6 & 3 \\ 10 & 5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow 2\xi_1 + \xi_2 = 0$$

Taking $\xi_1 = 1$, and eigen vector of $r = 0$ is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(b) An eigenvector foe $r = 11$ is given by $(\mathbf{A} - 11I_2)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 6-11 & 3 \\ 10 & 5-11 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow -5\xi_1 + 3\xi_2 = 0$$

Taking $\xi_1 = 3$, and eigen vector of $r = 11$ is

$$\xi^{(2)} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{11t}$$

(c) So, the general solution

$$y = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{11t}$$

2. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} -1 & 3 \\ 4 & 3 \end{pmatrix} \mathbf{y}$$

Solution First, the eigen values:

$$\begin{vmatrix} -1-r & 3 \\ 4 & 3-r \end{vmatrix} = 0 \quad \Longrightarrow \quad r^2 - 2r - 15 = 0 \Longrightarrow \quad r = -3, 5$$

(a) An eigenvector foe $r = -3$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} -1+3 & 3 \\ 4 & 3+3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow 2\xi_1 + 3\xi_2 = 0$$

Taking $\xi_1 = 3$, and eigen vector of $r = -3$ is

$$\xi^{(1)} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{-3t}$$

(b) An eigenvector foe $r = 5$ is given by $(\mathbf{A} - \lambda I_2)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} -1-5 & 3 \\ 4 & 3-5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow -6\xi_1 + 3\xi_2 = 0$$

Taking $\xi_1 = 1$, and eigen vector of $r = 5$ is

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$$

(c) So, the general solution

$$y = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} = c_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$$

3. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 3 & 8 \\ 2 & -3 \end{pmatrix} \mathbf{y}$$

Solution First, the eigen values:

$$\begin{vmatrix} 3-r & 8 \\ 2 & -3-r \end{vmatrix} = 0 \quad \Longrightarrow \quad r^2 - 25 = 0 \Longrightarrow \quad r = -5, 5$$

(a) An eigenvector foe $r = -5$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 3+5 & 8 \\ 2 & -3+5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 + \xi_2 = 0$$

Taking $\xi_1 = 1$, and eigen vector of $r = -5$ is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}$$

(b) An eigenvector foe $r = 5$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 3-5 & 8 \\ 2 & -3-5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow -2\xi_1 + 8\xi_2 = 0$$

Taking $\xi_2 = 1$, and eigen vector of $r = 5$ is

$$\xi^{(2)} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{5t}$$

(c) So, the general solution

$$y = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + c_2\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{5t}$$

4. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{y}$$

Solution First, the eigen values:

$$\begin{vmatrix} 2-r & 1 & -3 \\ 0 & -1-r & 1 \\ 0 & 1 & -1-r \end{vmatrix} = 0 \Longrightarrow (2-r)\begin{vmatrix} -1-r & 1 \\ 1 & -1-r \end{vmatrix} = 0 \Longrightarrow$$

$$(2-r)(r+2)r = 0 \Longrightarrow r = -2, 0, 2$$

(a) An eigenvector foe $r = -2$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 4 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 4 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Taking $\xi_2 = 1$, and eigen vector of $r = -2$ is

$$\xi^{(1)} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(1)} = \xi^{(1)}e^{rt} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^{-2t}$$

(b) An eigenvector foe $r = 0$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking $\xi_3 = 1$, and eigen vector of $r = 0$ is

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) An eigenvector foe $r = 2$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 2-2 & 1 & -3 \\ 0 & -1-2 & 1 \\ 0 & 1 & -1-2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & -3 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & -3 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \quad \xi_2 = \xi_3 = 0$$

Taking $\xi_1 = 1$, and eigen vector of $r = 2$ is

$$\xi^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(3)} = \xi^{(3)} e^{rt} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

(d) So, the general solution

$$y = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} + c_3 \mathbf{y}^{(2)}$$

$$= c_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

5. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix} \mathbf{y}$$

Solution First, the eigen values:

$$\begin{vmatrix} 4-r & 3 & -5 \\ 0 & -1-r & 3 \\ 0 & 3 & -1-r \end{vmatrix} = 0 \Longrightarrow (4-r)\begin{vmatrix} -1-r & 3 \\ 3 & -1-r \end{vmatrix} = 0 \Longrightarrow$$

$$(4-r)(r^2+2r-8) = 0 \Longrightarrow \qquad r = -4, 2, 4$$

(a) An eigenvector foe $r = -4$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 4+4 & 3 & -5 \\ 0 & -1+4 & 3 \\ 0 & 3 & -1+4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 8 & 3 & -5 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 8 & 3 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Taking $\xi_3 = 1$, and eigen vector of $r = -4$ is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(1)} = \xi^{(1)}e^{rt} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-4t}$$

(b) An eigenvector foe $r = 2$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 4-2 & 3 & -5 \\ 0 & -1-2 & 3 \\ 0 & 3 & -1-2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 2 & 3 & -5 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 2 & 3 & -5 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Taking $\xi_3 = 1$, and eigen vector of $r = 2$ is

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}$$

(c) An eigenvector foe $r = 4$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 4-4 & 3 & -5 \\ 0 & -1-4 & 3 \\ 0 & 3 & -1-4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 3 & -5 \\ 0 & -5 & 3 \\ 0 & 3 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 3 & -5 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_3 = \xi_2 = 0$$

Taking $\xi_1 = 1$, and eigen vector of $r = 4$ is

$$\xi^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(3)} = \xi^{(3)} e^{rt} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t}$$

(d) So, the general solution

$$y = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} + c_3 \mathbf{y}^{(2)}$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t}$$

5.6 Complex Eigenvalues

Consider homogeneous systems $\mathbf{y}' = A\mathbf{y}$, where A a constant matrix, of size $n \times n$. The section deals with complex values of the characteristic Equation $|A - \lambda I| = 0$. Before we proceed, recall from §5.6, corresponding a pari of conjugate eigen value $r = \lambda \pm i\mu$, and eigen vector $\xi = \mathbf{a} + i\mathbf{b}$, two solutions

$$\begin{cases} \mathbf{u} = e^{\lambda t} (\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) \\ \mathbf{v} = e^{\lambda t} (\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t) \end{cases} \tag{5.1}$$

1. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 2 & -1 \\ 13 & -2 \end{pmatrix} \mathbf{y}$$

Solution. First, the eigen values

$$\begin{vmatrix} 2-r & -1 \\ 13 & -2-r \end{vmatrix} = 0 \Longrightarrow r^2 + 9 = 0 \Longrightarrow r = \pm 3i$$

Now, an eigen vector of $r = 3i$, is given by

$$\begin{pmatrix} 2-3i & -1 \\ 13 & -2-3i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2-3i & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

With $\xi_1 = 1$, an eigen vector

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 2-3i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

So, two solutions, corresponding to $r = \pm 3i$ are (5.1):

$$\begin{cases} \mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t \\ \\ \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t \end{cases}$$

So, a general solution is:

$$\mathbf{y} = c_1 \mathbf{u} + c_2 \mathbf{v} =$$

$$c_1 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t \right) + c_2 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t \right)$$

2. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix} \mathbf{y}$$

Solution First, the eigen values

$$\begin{vmatrix} -1-r & -1 \\ 4 & -1-r \end{vmatrix} = 0 \Longrightarrow (r+1)^2 + 4 = 0 \Longrightarrow r = -1 \pm 2i$$

Now, an eigen vector of $r = -1 + 2i$, is given by

$$\begin{pmatrix} -1-(-1+2i) & -1 \\ 4 & -1-(-1+2i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -2i & -1 \\ 4 & 2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow -2i\xi_1 - \xi_2 = 0$$

With $\xi_1 = 1$, an eigen vector

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

So, two solutions, corresponding to $r = \pm 3i$ are (5.1):

$$\begin{cases} \mathbf{u} = e^{-t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t\right) = e^{-t}\begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} \\ \mathbf{v} = e^{-t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t\right) = e^{-t}\begin{pmatrix} \sin 2t \\ -2\cos 2t \end{pmatrix} \end{cases}$$

So, a general solution is:

$$\mathbf{y} = c_1\mathbf{u} + c_2\mathbf{v} = c_1 e^{-t}\begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin 2t \\ -2\cos 2t \end{pmatrix}$$

3. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} -1 & \pi \\ -\pi & -1 \end{pmatrix} \mathbf{y}$$

Solution
First, the eigen values

$$\begin{vmatrix} -1-r & \pi \\ -\pi & -1-r \end{vmatrix} = 0 \Longrightarrow (1+r)^2 + \pi^2 = r^2 + 2r + (1+\pi^2) = 0 \Longrightarrow$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1+\pi^2)}}{2} = -1 \pm \pi i$$

Now, an eigen vector of $r = -1 + \pi i$, is given by

$$\begin{pmatrix} -1-(-1+\pi i) & \pi \\ -\pi & -1-(-1+\pi i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -\pi i & \pi \\ -\pi & -\pi i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

With $\xi_1 = 1$, an eigen vector

$$\xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, two solutions, corresponding to $r = \pm 3i$ are (5.1):

$$\begin{cases} \mathbf{u} = e^{-t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos \pi t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin \pi t\right) = e^{-t}\begin{pmatrix} \cos \pi t \\ -\sin \pi t \end{pmatrix} \\ \\ \mathbf{v} = e^{-t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin \pi t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos \pi t\right) = e^{-t}\begin{pmatrix} \sin \pi t \\ \cos \pi t \end{pmatrix} \end{cases}$$

So, a general solution is:

$$\mathbf{y} = c_1 \mathbf{u} + c_2 \mathbf{v} = c_1 e^{-t} \begin{pmatrix} \cos \pi t \\ -\sin \pi t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin \pi t \\ \cos \pi t \end{pmatrix}$$

4. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} -1 & 7 & 0 \\ -7 & -1 & 0 \\ 3 & 4 & 4 \end{pmatrix} \mathbf{y}$$

Solution

First, the eigen values

$$\begin{vmatrix} -1-r & 7 & 0 \\ -7 & -1-r & 0 \\ 3 & 4 & 4-r \end{vmatrix} = 0 \Longrightarrow -(4-r)(r^2+2r+50) = 0 \Longrightarrow$$

$$r = 4, \frac{-2 \pm \sqrt{4-200}}{2} = 4, -1 \pm 7i$$

(a) Now, an eigen vector of $r = 4$, is given by

$$\begin{pmatrix} -1-4 & 7 & 0 \\ -7 & -1-4 & 0 \\ 3 & 4 & 4-4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -5 & 7 & 0 \\ -7 & -5 & 0 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\xi_1 = \xi_2 = 0 \quad \text{Taking } \xi_3 = 1 \text{ an eigen vector } \xi^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Correspondingly, a solution

$$\mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t}$$

(b) Now, an eigen vector of $r = -1 + 7i$, is given by

$$\begin{pmatrix} -1-(-1+7i) & 7 & 0 \\ -7 & -1-(-1+7i) & 0 \\ 3 & 4 & 4-(-1+7i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -7i & 7 & 0 \\ -7 & -7i & 0 \\ 3 & 4 & 5-7i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \\ 3 & 4 & 5-7i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \\ 0 & 4-3i & 5-7i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Taking $\xi_2 = 1$, we have $\xi_1 = -i$ and
$(4-3i) + (5-7i)\xi_3 = 0 \Longrightarrow$

$$\xi_3 = -\frac{4-3i}{5-7i} = -\frac{(4-3i)(5+7i)}{74} = -\frac{41+13i}{74}$$

So, an eigen vector

$$\xi^{(2)} = \begin{pmatrix} -i \\ 1 \\ -\frac{41+13i}{74} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -\frac{41}{74} \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \\ -\frac{13}{74} \end{pmatrix}$$

By (5.1) two solutions:

$$\begin{cases} \mathbf{u} = e^{-t}\left(\begin{pmatrix} 0 \\ 1 \\ -\frac{41}{74} \end{pmatrix} \cos 7t - \begin{pmatrix} -1 \\ 0 \\ -\frac{13}{74} \end{pmatrix} \sin 7t\right) = e^{-t}\begin{pmatrix} \sin 7t \\ \cos 7t \\ -\frac{41}{74}\cos 7t + \frac{13}{74}\sin 7t \end{pmatrix} \\ \\ \mathbf{v} = e^{-t}\left(\begin{pmatrix} 0 \\ 1 \\ -\frac{41}{74} \end{pmatrix} \sin 7t + \begin{pmatrix} -1 \\ 0 \\ -\frac{13}{74} \end{pmatrix} \cos 7t\right) = e^{-t}\begin{pmatrix} -\cos 7t \\ \sin 7t \\ -\frac{41}{74}\sin 7t - \frac{13}{74}\cos 7t \end{pmatrix} \end{cases}$$

So, a general solution is

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{u} + c_3\mathbf{v} =$$

$$c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t} + c_2 e^{-t} \begin{pmatrix} \sin 7t \\ \cos 7t \\ -\frac{41}{74}\cos 7t + \frac{13}{74}\sin 7t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} -\cos 7t \\ \sin 7t \\ -\frac{41}{74}\sin 7t - \frac{13}{74}\cos 7t \end{pmatrix}$$

5. Find a general solutions of

$$\mathbf{y}' = \begin{pmatrix} 5 & 0 & 9 \\ 0 & \pi & 0 \\ -4 & 0 & -5 \end{pmatrix} \mathbf{y}$$

Solution
First, the eigen values

$$\begin{vmatrix} 5-r & 0 & 9 \\ 0 & \pi - r & 0 \\ -4 & 0 & -5-r \end{vmatrix} = 0 \Longrightarrow (\pi-r)(r^2-25)+36(\pi-r) = 0 \Longrightarrow (\pi-r)(r^2+11) = 0$$

$$r = \pi, \pm\sqrt{11}i$$

(a) Now, an eigen vector of $r = \pi$, is given by

$$\begin{pmatrix} 5-\pi & 0 & 9 \\ 0 & \pi-\pi & 0 \\ -4 & 0 & -5-\pi \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 = \xi_3 = 0$$

$$\text{Taking } \xi_2 = 1 \text{ an eigen vector } \xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Correspondingly, a solution } \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{\pi t}$$

(b) Now, an eigen vector of $r = \sqrt{11}i$, is given by

$$\begin{pmatrix} 5-\sqrt{11}i & 0 & 9 \\ 0 & \pi-\sqrt{11}i & 0 \\ -4 & 0 & -5-\sqrt{11}i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{cases} (5-\sqrt{11}i)\xi_1 + 9\xi_3 = 0 \\ \xi_2 = 0 \\ -4\xi_1 - (5+\sqrt{11}i)\xi_3 = 0 \end{cases} \Longrightarrow \begin{cases} (5-\sqrt{11}i)\xi_1 + 9\xi_3 = 0 \\ \xi_2 = 0 \\ 0 = 0 \end{cases}$$

Taking $\xi_1 = -9$, $\xi_3 = 5 - \sqrt{11}i$. So and eigenvalue for $r = \sqrt{11}i$ is

$$\xi^{(2)} = \begin{pmatrix} -9 \\ 0 \\ 5 - \sqrt{11}i \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ 5 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ -\sqrt{11} \end{pmatrix}$$

By (5.1) two solutions:

$$\begin{cases} \mathbf{u} = \begin{pmatrix} -9 \\ 0 \\ 5 \end{pmatrix} \cos\sqrt{11}t - \begin{pmatrix} 0 \\ 0 \\ -\sqrt{11} \end{pmatrix} \sin\sqrt{11}t = \begin{pmatrix} -9\cos\sqrt{11}t \\ 0 \\ 5\cos\sqrt{11}t + \sqrt{11}\sin\sqrt{11}t \end{pmatrix} \\ \\ \mathbf{v} = \begin{pmatrix} -9 \\ 0 \\ 5 \end{pmatrix} \sin\sqrt{11}t + \begin{pmatrix} 0 \\ 0 \\ -\sqrt{11} \end{pmatrix} \cos\sqrt{11}t = \begin{pmatrix} 9\sin 3t \\ 0 \\ 5\sin 3t - 3\cos 3t \end{pmatrix} \end{cases}$$

(c) So, a general solution is

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{u} + c_3\mathbf{v} =$$

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{\pi t} + c_2 \begin{pmatrix} -9\cos\sqrt{11}t \\ 0 \\ 5\cos\sqrt{11}t + \sqrt{11}\sin\sqrt{11}t \end{pmatrix} + c_3 \begin{pmatrix} 9\sin 3t \\ 0 \\ 5\sin 3t - 3\cos 3t \end{pmatrix}$$

5.7 Repeated Egenvalues

1. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} \mathbf{y}$$

Solution
First the eigen values:

$$\begin{vmatrix} 5-r & -1 \\ 4 & 1-r \end{vmatrix} = 0 \Longrightarrow r^2 - 6r + 9 = 0 \Longrightarrow r = 3 \quad \text{is a double eigen value.}$$

(a) Eigen vector for $r = 3$ and the corresponding solution:

$$\begin{pmatrix} 5-3 & -1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow 2\xi_1 - \xi_2 = 0$$

Taking $\xi_1 = 1$, and eigen vector for $r = 3$:

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{The corresponding solution} \quad \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

(b) To compute the second solution, solve $(A - rI)\eta = \xi^{(1)}$. So,

$$\begin{pmatrix} 5-3 & -1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow$$

$$2\eta_1 - \eta_2 = 1. \quad \text{Taking } \eta_2 = 1 \ \textit{we have} \quad \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So, a second solution is

$$\mathbf{y}^{(2)} = \xi^{(1)} t e^{rt} + \eta e^{rt} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t e^{3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} t+1 \\ 2t+1 \end{pmatrix} e^{3t}$$

(c) So, a general solution:

$$y = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} t+1 \\ 2t+1 \end{pmatrix} e^{3t}$$

2. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} 3 & -5 \\ 5 & -7 \end{pmatrix} \mathbf{y}$$

Solution

First the eigen values:

$$\begin{vmatrix} 3-r & -5 \\ 5 & -7-r \end{vmatrix} = 0 \Longrightarrow r^2 + 4r + 4 = 0 \Longrightarrow r = -2 \quad \text{is a double eigen value.}$$

(a) Eigen vector for $r = -2$ and the corresponding solution:

$$\begin{pmatrix} 3+2 & -5 \\ 5 & -7+2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 - \xi_2 = 0$$

Taking $\xi_1 = 1$, and eigen vector for $r = -2$:

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{The corresponding solution} \quad \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$$

(b) To compute the second solution, solve $(A - rI)\eta = \xi^{(1)}$. So,

$$\begin{pmatrix} 3+2 & -5 \\ 5 & -7+2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 5 & -5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow$$

$$5\eta_1 - 5\eta_2 = 1. \quad \text{Taking } \eta_2 = 0 \ \textit{we have} \quad \eta = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}$$

So, a second solution is

$$\mathbf{y}^{(2)} = \xi^{(1)} t e^{rt} + \eta e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} e^{-2t} = \begin{pmatrix} t + \frac{1}{5} \\ t \end{pmatrix} e^{-2t}$$

(c) So, a general solution:

$$y = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} t + \frac{1}{5} \\ t \end{pmatrix} e^{-2t}$$

3. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} \pi & -\pi \\ \pi & 3\pi \end{pmatrix} \mathbf{y}$$

Solution
First the eigen values:

$$\begin{vmatrix} \pi - r & -\pi \\ \pi & 3\pi - r \end{vmatrix} = 0 \Longrightarrow r^2 - 4\pi r + 4\pi^2 = 0 \Longrightarrow r = 2\pi \quad \text{is a double eigen value.}$$

(a) Eigen vector for $r = 2\pi$ and the corresponding solution:

$$\begin{pmatrix} \pi - 2\pi & -\pi \\ \pi & 3\pi - 2\pi \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 + \xi_2 = 0$$

Taking $\xi_1 = 1$, and eigen vector for $r = 3$:

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{The corresponding solution} \quad \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2\pi t}$$

(b) To compute the second solution, solve $(A - rI)\eta = \xi^{(1)}$. So,

$$\begin{pmatrix} \pi - 2\pi & -\pi \\ \pi & 3\pi - 2\pi \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} -\pi & -\pi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow$$

$$-\pi\eta_1 - \pi\eta_2 = 1. \quad \text{Taking } \eta_2 = 0 \ \ \textit{we have} \quad \eta = \begin{pmatrix} -\frac{1}{\pi} \\ 0 \end{pmatrix}$$

So, a second solution is

$$\mathbf{y}^{(2)} = \xi^{(1)} t e^{rt} + \eta e^{rt} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{\pi t} + \begin{pmatrix} -\frac{1}{\pi} \\ 0 \end{pmatrix} e^{\pi t} = \begin{pmatrix} t - \frac{1}{\pi} \\ -t \end{pmatrix} e^{\pi t}$$

(c) So, a general solution:

$$y = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2\pi t} + c_2 \begin{pmatrix} t - \frac{1}{\pi} \\ -t \end{pmatrix} e^{\pi t}$$

4. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \mathbf{y}$$

Help: There would be two linearly independent eigen vector, for the double eigen value.

Solution

First the eigen values:

$$\begin{vmatrix} 3 - r & 0 & -1 \\ 0 & 2 - r & 0 \\ -1 & 0 & 3 - r \end{vmatrix} = 0 \Longrightarrow (3 - r)^2(2 - r) - (2 - r) = 0 \quad \Longrightarrow$$

$$-(2-r)^2(r-4)=0 \Longrightarrow \quad r=\begin{cases} 4 & \text{a simple E.V.} \\ 2 & \text{a double E.V.} \end{cases}$$

(a) An eigen vector for $r=4$ and a solution:
Since $r=4$ is a simple E. V., we expect only one linearly independent eigen vector:

$$\begin{pmatrix} 3-4 & 0 & -1 \\ 0 & 2-4 & 0 \\ -1 & 0 & 3-4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 2-4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} \xi_1+\xi_3=0 \\ \xi_2=0 \end{cases}$$

Taking $\xi_1=1$, and eigen vector for $r=4$:

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{The corresponding solution} \quad \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{4t}$$

(b) An eigen vector for $r=2$ and solution(s):
Since $r=2$ is a double E. V., we may get one or two linearly independent eigen vector:

$$\begin{pmatrix} 3-2 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 3-2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \{ \ \xi_1-\xi_3=0$$

In fact, we get two linearly independent eigen vectors:

$$\text{with } \xi_1=1, \xi_2=0 \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \text{ and, with } \xi_1=0, \xi_2=1 \quad \xi^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The corresponding solutions:

$$\mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}, \quad \mathbf{y}^{(3)} = \xi^{(3)} e^{rt} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

(c) So, a general solution

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + c_3\mathbf{y}^{(3)}$$

$$= c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{4t} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

5. Find the general solution of the system of ODE

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ -1 & -1 & 1 \end{pmatrix} \mathbf{y}$$

Help: There would be Only ONE linearly independent eigen vector, for the double eigen value.

Solution
First the eigen values:

$$\begin{vmatrix} 2-r & 1 & 1 \\ 1 & -r & -2 \\ -1 & -1 & 1-r \end{vmatrix} = 0 \Longrightarrow$$

$$(2-r)\begin{vmatrix} -r & -2 \\ -1 & 1-r \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ -1 & 1-r \end{vmatrix} + \begin{vmatrix} 1 & -r \\ -1 & -1 \end{vmatrix} = 0 \Longrightarrow$$

$$(2-r)(r^2-r-2) = 0 \Longrightarrow -(r-2)^2(r+1) = 0 \Longrightarrow r = \begin{cases} -1 & \text{a simple E.V.} \\ 2 & \text{a double E.V.} \end{cases}$$

(a) An eigen vector for $r = -1$ and a solution:
Since $r = -1$ is a simple E. V., we expect only one linearly independent eigen vector:

$$\begin{pmatrix} 2+1 & 1 & 1 \\ 1 & 1 & -2 \\ -1 & -1 & 1+1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & -2 & 7 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} -2\xi_2 + 7\xi_3 = 0 \\ \xi_1 + \xi_2 - 2\xi_3 = 0 \end{cases}$$

Taking $\xi_3 = 2$ and eigen vector of $r = -1$ is

$$\xi^{(1)} = \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} \text{ The corresponding solution } \quad \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} e^{-t}$$

(b) An eigen vector for $r = 2$ and solution(s):
Since $r = 2$ is a double E. V., we may get one or two linearly independent eigen vector:

$$\begin{pmatrix} 2-2 & 1 & 1 \\ 1 & -2 & -2 \\ -1 & -1 & 1-2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} \xi_2 + \xi_3 = 0 \\ \xi_1 = 0 \end{cases}$$

$$\text{With } \xi_3 = 1 \quad \text{An E.V. } \xi^{(2)} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

An an solution

$$\mathbf{y}^{(2)} = \xi^{(2)} r^{rt} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t}$$

(c) A Third Solution: Since we got only one E. Vector. for the double eigen value $r = 2$, to compute a third solution, we solve $(A - \lambda I)\eta = \xi^{(2)}$:

$$\begin{pmatrix} 2-2 & 1 & 1 \\ 1 & -2 & -2 \\ -1 & -1 & 1-2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Longrightarrow \begin{cases} \eta_2 + \eta_3 = 0 \\ \eta_1 = -1 \\ -\eta_1 = 1 \end{cases}$$

Taking $\eta_2 = 1$, we have

$$\eta = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

So, a third solution

$$\mathbf{y}^{(3)} = \xi^{(2)} t e^{rt} + \eta e^{rt} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t e^{2t} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^{2t} = \begin{pmatrix} -1 \\ -t+1 \\ t-1 \end{pmatrix} e^{2t}$$

(d) So, a general solution

$$y = c_1 \mathbf{y}^{(1)} + \mathbf{y}^{(2)} + c_3 \mathbf{y}^{(3)}$$

$$= c_1 \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} -1 \\ -t+1 \\ t-1 \end{pmatrix} e^{2t}$$

5.8 Nonhomogeneous Linear Systems

For the purpose of this course, we consider problems in this sections, so that the respective eigen values are real and distinct.
(Please double check for possible numerical errors.)

1. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$$

   Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

Solution. First, the eigen values:

$$\begin{vmatrix} 2-r & 4 \\ 3 & -2-r \end{vmatrix} = 0 \Longrightarrow r^2 - 16 = 0 \Longrightarrow r = -4, 4$$

An eigen vector for $r = -4$ is given by

$$\begin{pmatrix} 2+4 & 4 \\ 3 & -2+4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Taking $\xi_1 = 2$, an eigen vector for $r = -4$ is

$$\xi^{(1)} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

The corresponding solution of the homogeneous System $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-4t}$$

An eigen vector for $r = 4$ is given by

$$\begin{pmatrix} 2-4 & 4 \\ 3 & -2-4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Taking $\xi_2 = 1$, an eigen vector for $r = 4$ is

$$\xi^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The corresponding solution of the homogeneous System $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t}$$

(a) So, the matrix of the eigen vectors:

$$T = \begin{pmatrix} \xi^{(1)} & \xi^{(2)} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}$$

$$T^{-1} = \frac{1}{8}\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$$

(b) With $\mathbf{z} = T^{-1}\mathbf{y}$, we have

$$\mathbf{z}' = T^{-1}\mathbf{y}' = T^{-1}\begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}\mathbf{y} + T^{-1}\begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$$

$$= T^{-1}\begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}T\mathbf{z} + \frac{1}{8}\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} \Longrightarrow$$

$$\mathbf{z}' = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{z} + \begin{pmatrix} -\frac{1}{8}e^{-2t} \\ \frac{5}{8}e^{-2t} \end{pmatrix} \Longrightarrow \begin{cases} z_1' + 4z_1 = -\frac{1}{8}e^{-2t} \\ z_2' - 4z_2 = \frac{5}{8}e^{-2t} \end{cases}$$

(c) Now we solve for z_1 and z_2.

i. To solve for z_1, the IF $\mu(t) = \exp(4dt) = e^{4t}$. So, a particular solution

$$Z_1 = \frac{1}{\mu(t)}\int \mu(t)h_1(t)dt = -\frac{1}{8}e^{-4t}\int e^{4t}e^{-2t}dt = -\frac{1}{8}e^{-4t}\frac{e^{2t}}{2} = -\frac{1}{16}e^{-2t}$$

ii. To solve for z_2, the IF $\mu(t) = \exp(-4dt) = e^{-4t}$. So, a particular solution

$$Z_2 = \frac{1}{\mu(t)}\int \mu(t)h_2(t)dt = \frac{5}{8}e^{4t}\int e^{-4t}e^{-2t}dt = \frac{5}{8}e^{4t}\frac{e^{-6t}}{-6} = -\frac{5}{48}e^{-2t}$$

So, a particular solution:

$$\mathbf{z} = \mathbf{Z} = \begin{pmatrix} -\frac{1}{16}e^{-2t} \\ -\frac{5}{48}e^{-2t} \end{pmatrix}$$

(d) So, a particular solution for $\mathbf{y}$ is

$$\mathbf{Y} = T\mathbf{Z} = \begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{16}e^{-2t} \\ -\frac{5}{48}e^{-2t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{12} \end{pmatrix}e^{-2t}$$

So, a general solution for $\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \mathbf{Y} = c_1\begin{pmatrix} 2 \\ -3 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{4t} + \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{12} \end{pmatrix}e^{-2t}$$

2. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} t \\ e^{-2t} \end{pmatrix}$$

Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

Solution. First, the eigen values:

$$\begin{vmatrix} -r & 2 \\ 3 & -1-r \end{vmatrix} = 0 \Longrightarrow r^2 + r - 6 = 0 \Longrightarrow r = -3, 2$$

An eigen vector for $r = -3$ is given by

$$\begin{pmatrix} 3 & 2 \\ 3 & -1+3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow 3\xi_1 + 2\xi_2 = 0$$

Taking $\xi_1 = 2$, an eigen vector for $r = -3$ is

$$\xi^{(1)} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

The corresponding solution of the homogeneous System $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-3t}$$

An eigen vector for $r = 2$ is given by

$$\begin{pmatrix} -2 & 2 \\ 3 & -1-2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Taking $\xi_1 = 1$, an eigen vector for $r = 2$ is

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The corresponding solution of the homogeneous System $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

(a) So, the matrix of the eigen vectors:

$$T = \begin{pmatrix} \xi^{(1)} & \xi^{(2)} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$$

$$T^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

(b) With $\mathbf{z} = T^{-1}\mathbf{y}$, we have

$$\mathbf{z}' = T^{-1}\mathbf{y}' = T^{-1}AT\mathbf{z} + \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} t \\ e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{z} + \frac{1}{5} \begin{pmatrix} t - e^{-2t} \\ 3t + 2e^{-2t} \end{pmatrix} \Longrightarrow \begin{cases} z_1' + 3z_1 = \frac{1}{5}\left(t - e^{-2t}\right) \\ z_2' - 2z_2 = \frac{1}{5}\left(3t + 2e^{-2t}\right) \end{cases}$$

(c) Now we solve for z_1 and z_2.

i. To solve for z_1, the IF $\mu(t) = \exp(3dt) = e^{3t}$. So, a particular solution

$$Z_1 = \frac{1}{\mu(t)} \int \mu(t) h_1(t) dt = \frac{1}{5} e^{-3t} \int e^{3t}(t - e^{-2t}) dt =$$

$$= \frac{1}{5} e^{-3t} \left(\frac{1}{3} \left(te^{3t} - \frac{e^{3t}}{3} \right) - e^t \right) = \frac{1}{5} t - \frac{1}{45} - \frac{1}{5} e^{-2t}$$

ii. To solve for z_2, the IF $\mu(t) = \exp(-2dt) = e^{-2t}$. So, a particular solution

$$Z_2 = \frac{1}{\mu(t)} \int \mu(t) h_2(t) dt = \frac{1}{5} e^{2t} \int e^{-2t} \left(3t + 2e^{-2t}\right) dt$$

$$= \frac{1}{5} e^{2t} \left(-\frac{3}{2} \left(te^{-2t} + \frac{e^{-2t}}{2} \right) - \frac{e^{-4t}}{2} \right) = -\frac{3}{10} t - \frac{3}{20} - \frac{1}{10} e^{-4t}$$

So, a particular solution:

$$\mathbf{z} = \mathbf{Z} = \begin{pmatrix} \frac{1}{5}t - \frac{1}{45} - \frac{1}{5}e^{-2t} \\ -\frac{3}{10}t - \frac{3}{20} - \frac{1}{10}e^{-4t} \end{pmatrix}$$

(d) So, a particular solution for $\mathbf{y}$ is

$$\mathbf{Y} = T\mathbf{Z} = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}t - \frac{1}{45} - \frac{1}{5}e^{-2t} \\ -\frac{3}{10}t - \frac{3}{20} - \frac{1}{10}e^{-4t} \end{pmatrix}$$

I would leave it in this matrix form.

So, a general solution for $\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \mathbf{Y} = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \mathbf{Y}$$

3. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} t \\ 2t \end{pmatrix}$$

Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

Solution. First, the eigen values:

$$\begin{vmatrix} 4-r & 3 \\ 1 & 2-r \end{vmatrix} = 0 \Longrightarrow r^2 - 6r + 5 \Longrightarrow r = 1, 5$$

An eigen vector for $r = 1$ is given by

$$\begin{pmatrix} 4-1 & 3 \\ 1 & 2-1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 + \xi_2 = 0$$

Taking $\xi_1 = 1$, an eigen vector for $r = 1$ is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The corresponding solution of the homogeneous System $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$$

An eigen vector for $r = 5$ is given by

$$\begin{pmatrix} 4-5 & 3 \\ 1 & 2-5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 - 3\xi_2 = 0$$

Taking $\xi_2 = 1$, an eigen vector for $r = 5$ is

$$\xi^{(2)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The corresponding solution of the homogeneous System $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t}$$

(a) So, the matrix of the eigen vectors:

$$T = \begin{pmatrix} \xi^{(1)} & \xi^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$$

$$T^{-1} = \frac{1}{4} \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$$

(b) With $\mathbf{z} = T^{-1}\mathbf{y}$, we have

$$\mathbf{z}' = T^{-1}\mathbf{y}' = T^{-1}AT\mathbf{z} + \frac{1}{4} \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{z} + \begin{pmatrix} -\frac{5}{4}t \\ \frac{3}{4}t \end{pmatrix} \Longrightarrow \begin{cases} z_1' - z_1 = -\frac{5}{4}t \\ z_2' - 5z_2 = \frac{3}{4}t \end{cases}$$

(c) Now we solve for z_1 and z_2.

i. To solve for z_1, the IF $\mu(t) = \exp\left(-dt\right) = e^{-t}$. So, a particular solution

$$Z_1 = \frac{1}{\mu(t)} \int \mu(t) h_1(t) dt = -\frac{5}{4} e^t \int t e^{-t} dt$$

$$= \frac{5}{4} e^t \left(t e^{-t} + e^{-t} \right) = \frac{5}{4}(t+1)$$

ii. To solve for z_2, the IF $\mu(t) = \exp(-5dt) = e^{-5t}$. So, a particular solution

$$Z_2 = \frac{1}{\mu(t)} \int \mu(t) h_2(t) dt = \frac{3}{4} e^{-5t} \int e^{5t} t dt$$

$$= \frac{3}{20} e^{-5t} \left(te^{5t} - \frac{e^{5t}}{5} \right) = \frac{3}{100}(5t-1)$$

So, a particular solution:

$$\mathbf{z} = \mathbf{Z} = \begin{pmatrix} \frac{5}{4}(t+1) \\ \frac{3}{100}(5t-1) \end{pmatrix}$$

(d) So, a particular solution for $\mathbf{y}$ is

$$\mathbf{Y} = T\mathbf{Z} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{4}(t+1) \\ \frac{3}{100}(5t-1) \end{pmatrix}$$

I would leave it in this matrix form.

So, a general solution for $\mathbf{y}$ is

$$\mathbf{y} = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} + \mathbf{Y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t} + \mathbf{Y}$$

4. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} \cos t \\ \cos t \end{pmatrix}$$

Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

Solution. First, the eigen values:

$$\begin{vmatrix} 1-r & \sqrt{2} \\ -\sqrt{2} & -2-r \end{vmatrix} = 0 \Longrightarrow r^2 + r \Longrightarrow r = -1, 0$$

An eigen vector for $r = -1$ is given by

$$\begin{pmatrix} 1+1 & \sqrt{2} \\ -\sqrt{2} & -2+1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$2\xi_1 + \sqrt{2}\xi_2 = 0. \quad \text{With } \xi_1 = 1 \text{ an Eigen vector } \ \xi^{(1)} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

The corresponding solution of the homogeneous System $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{-t}$$

An eigen vector for $r = 0$ is given by

$$\begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_1 + \sqrt{2}\xi_2 = 0$$

Taking $\xi_2 = 1$, an eigen vector for $r = 0$ is

$$\xi^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

The corresponding solution of the homogeneous System $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

(a) So, the matrix of the eigen vectors:

$$T = \begin{pmatrix} \xi^{(1)} & \xi^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$$

$$T^{-1} = -\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

(b) With $\mathbf{z} = T^{-1}\mathbf{y}$, we have

$$\mathbf{z}' = T^{-1}\mathbf{y}' = T^{-1}AT\mathbf{z} - \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \cos t \\ \cos t \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{z} + \begin{pmatrix} -(1+\sqrt{2})\cos t \\ -(1+\sqrt{2})\cos t \end{pmatrix} \Longrightarrow \begin{cases} z_1' + z_1 = -(1+\sqrt{2})\cos t \\ z_2' = -(1+\sqrt{2})\cos t \end{cases}$$

(c) Now we solve for z_1 and z_2.

i. To solve for z_1, the IF $\mu(t) = \exp(dt) = e^t$. So, a particular solution

$$Z_1 = \frac{1}{\mu(t)} \int \mu(t) h_1(t) dt = -(1+\sqrt{2})e^{-t} \int e^t \cos t dt$$

By (A.1.1)

$$Z_1 = -(1+\sqrt{2})e^{-t} \left(e^t \frac{\sin t + \cos t}{2} \right) = -\frac{(1+\sqrt{2})(\sin t + \cos t)}{2}$$

ii. Solve for $z_2 = Z_2$, directly

$$Z_2 = -(1+\sqrt{2}) \int \cos t dt = -(1+\sqrt{2}) \sin t$$

So, a particular solution:

$$\mathbf{z} = \mathbf{Z} = \begin{pmatrix} -\frac{(1+\sqrt{2})(\sin t + \cos t)}{2} \\ -(1+\sqrt{2}) \sin t \end{pmatrix} = -(1+\sqrt{2}) \begin{pmatrix} \frac{\sin t + \cos t}{2} \\ \sin t \end{pmatrix}$$

(d) So, a particular solution for $\mathbf{y}$ is

$$\mathbf{Y} = T\mathbf{Z} = -(1+\sqrt{2}) \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{\sin t + \cos t}{2} \\ \sin t \end{pmatrix}$$

I would leave it in this matrix form.

So, a general solution for $\mathbf{y}$ is

$$\mathbf{y} = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} + \mathbf{Y} = c_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + \mathbf{Y}$$

5. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -e^{2t} \\ -e^{2t} \\ e^{2t} \end{pmatrix}$$

Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

Solution First, the eigen values:

$$\begin{vmatrix} 2-r & 1 & -3 \\ 0 & -1-r & 1 \\ 0 & 1 & -1-r \end{vmatrix} = 0 \Longrightarrow (2-r)\begin{vmatrix} -1-r & 1 \\ 1 & -1-r \end{vmatrix} = 0 \Longrightarrow$$

$$(2-r)(r+2)r = 0 \Longrightarrow r = -2, 0, 2$$

An eigenvector foe $r = -2$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 4 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 4 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Taking $\xi_2 = 1$, and eigen vector of $r = -2$ is

$$\xi^{(1)} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(1)} = \xi^{(1)}e^{rt} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^{-2t}$$

An eigenvector foe $r = 0$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking $\xi_3 = 1$, and eigen vector of $r = 0$ is

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

An eigenvector foe $r = 2$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 2-2 & 1 & -3 \\ 0 & -1-2 & 1 \\ 0 & 1 & -1-2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & -3 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & -3 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \quad \xi_2 = \xi_3 = 0$$

Taking $\xi_1 = 1$, and eigen vector of $r = 2$ is

$$\xi^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(3)} = \xi^{(3)} e^{rt} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

(a) So, the matrix of the eigen vectors

$$T = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \Longrightarrow T^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

(b) With $\mathbf{z} = T^{-1}\mathbf{y}$, we have

$$\mathbf{z}' = T^{-1}\mathbf{y}' = T^{-1}AT\mathbf{z} + \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ -e^{2t} \\ e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{z} + \begin{pmatrix} -e^{2t} \\ 0 \\ -2e^{2t} \end{pmatrix} \Longrightarrow \begin{cases} z_1' + 2z_1 = -e^{2t} \\ z_2' = 0 \\ z_3' - 2z_3' = -2e^{2t} \end{cases}$$

(c) Now we solve for z_1, z_2 and z_3.

i. To solve for z_1, the IF $\mu(t) = \exp\left(\int 2dt\right) = e^{2t}$. So, a particular solution

$$Z_1 = \frac{1}{\mu(t)} \int \mu(t)h_1(t)dt = -e^{-2t} \int e^{2t}e^{2t}dt = -e^{-2t}\frac{e^{4t}}{4} = -\frac{e^{2t}}{4}$$

ii. Also,

$$Z_2 = 1$$

iii. To solve for z_3, the IF $\mu(t) = \exp\left(\int -2dt\right) = e^{-2t}$. So, a particular solution

$$Z_3 = \frac{1}{\mu(t)} \int \mu(t)h_3(t)dt = -2e^{2t} \int e^{-2t}(e^{2t})dt = -2te^{2t}$$

So, a particular solution:

$$\mathbf{z} = \mathbf{Z} = \begin{pmatrix} -\frac{e^{2t}}{4} \\ 1 \\ -2te^{2t} \end{pmatrix}$$

(d) So, a particular solution for $\mathbf{y}$ is

$$\mathbf{Y} = T\mathbf{Z} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{e^{2t}}{4} \\ 1 \\ -2te^{2t} \end{pmatrix}$$

I would leave it in this matrix form.

So, a general solution for $\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + c_3\mathbf{y}^{(3)} + \mathbf{Y}$$
$$= c_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + \mathbf{Y}$$

6. Give a general solution of the nonhomogeneous system,

$$\mathbf{y}' = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ 2 \\ 2t \end{pmatrix}$$

Make sure to show the following steps:

(a) Compute the matrix T of eigen vectors.

(b) Do the change of variables $\mathbf{z} = T^{-1}\mathbf{y}$.

(c) Compute a particular solution $\mathbf{z} = \mathbf{Z}$.

(d) Write down a general solution for $\mathbf{y}$

Solution First, the eigen values:

$$\begin{vmatrix} 4-r & 3 & -5 \\ 0 & -1-r & 3 \\ 0 & 3 & -1-r \end{vmatrix} = 0 \Longrightarrow (4-r)\begin{vmatrix} -1-r & 3 \\ 3 & -1-r \end{vmatrix} = 0 \Longrightarrow$$

$$(4-r)(r^2+2r-8) = 0 \Longrightarrow \qquad r = -4, 2, 4$$

An eigenvector for $r = -4$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 4+4 & 3 & -5 \\ 0 & -1+4 & 3 \\ 0 & 3 & -1+4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 8 & 3 & -5 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 8 & 3 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Taking $\xi_3 = 1$, and eigen vector of $r = -4$ is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(1)} = \xi^{(1)}e^{rt} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-4t}$$

An eigenvector for $r = 2$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 4-2 & 3 & -5 \\ 0 & -1-2 & 3 \\ 0 & 3 & -1-2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 2 & 3 & -5 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 2 & 3 & -5 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Taking $\xi_3 = 1$, and eigen vector of $r = 2$ is

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}$$

An eigenvector for $r = 4$ is given by $(\mathbf{A} - \lambda I)\xi = \mathbf{0}$. So,

$$\begin{pmatrix} 4-4 & 3 & -5 \\ 0 & -1-4 & 3 \\ 0 & 3 & -1-4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 3 & -5 \\ 0 & -5 & 3 \\ 0 & 3 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 3 & -5 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \xi_3 = \xi_2 = 0$$

Taking $\xi_1 = 1$, and eigen vector of $r = 4$ is

$$\xi^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad \text{So, a solution} \quad \mathbf{y}^{(3)} = \xi^{(3)} e^{rt} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t}$$

(a) So, the matrix of the eigen vectors

$$T = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \Longrightarrow T^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

(b) With $\mathbf{z} = T^{-1}\mathbf{y}$, we have

$$\mathbf{z}' = T^{-1}\mathbf{y}' = T^{-1}AT\mathbf{z} + \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\mathbf{z} + \begin{pmatrix} 1-t \\ 1+t \\ -t \end{pmatrix} \Longrightarrow \begin{cases} z_1' + 4z_1 = 1-t \\ z_2' - 2z = 1+t \\ z_3' - 4z_3' = -t \end{cases}$$

(c) Now we solve for z_1, z_2 and z_3.

i. To solve for z_1, the IF $\mu(t) = \exp\left(\int 4dt\right) = e^{4t}$. So, a particular solution

$$Z_1 = \frac{1}{\mu(t)}\int \mu(t)h_1(t)dt = e^{-4t}\int e^{4t}(1-t)dt = e^{-4t}\left(\frac{e^{4t}}{4} - \frac{1}{4}\left(te^{4t} - \frac{e^{4t}}{4}\right)\right)$$

$$= \frac{1}{2} - \frac{1}{4}t = \frac{2t-1}{4}$$

ii. To solve for z_2, the IF $\mu(t) = \exp\left(\int -2dt\right) = e^{-2t}$. So, a particular solution

$$Z_2 = \frac{1}{\mu(t)}\int \mu(t)h_1(t)dt = e^{2t}\int e^{-2t}(1+t)dt = e^{2t}\left(\frac{-e^{-2t}}{2} - \frac{1}{2}\left(te^{-2t} + \frac{e^{-2t}}{2}\right)\right)$$

$$= -\frac{1}{2} - \frac{1}{2}t = -\frac{1+t}{2}$$

iii. To solve for z_3, the IF $\mu(t) = \exp\left(\int -4dt\right) = e^{-4t}$. So, a particular solution

$$Z_3 = \frac{1}{\mu(t)}\int \mu(t)h_1(t)dt = -e^{4t}\int e^{-4t}tdt = \frac{1}{4}e^{4t}\left(te^{-4t} + \frac{e^{-4t}}{4}\right) = \frac{4t+1}{16}$$

So, a particular solution:

$$\mathbf{z} = \mathbf{Z} = \begin{pmatrix} \frac{2t-1}{4} \\ -\frac{1+t}{2} \\ \frac{4t+1}{16} \end{pmatrix}$$

(d) So, a particular solution for $\mathbf{y}$ is

$$\mathbf{Y} = T\mathbf{Z} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2t-1}{4} \\ -\frac{1+t}{2} \\ \frac{4t+1}{16} \end{pmatrix}$$

I would leave it in this matrix form.

So, a general solution for $\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + c_3\mathbf{y}^{(3)} + \mathbf{Y}$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t} + \mathbf{Y}$$

Chapter 6

The Laplace Transform

6.1 Definition of Laplace Transform

1. Compute the Laplace Transform of the function $f(t) = t$, from the definition. (That means, do not use the Charts.)

   Solution

   $$\mathcal{L}\{t\}(s) = \int_0^\infty te^{-st}dt = -\frac{1}{s}\int_0^\infty tde^{-st} = -\frac{1}{s}\left[te^{-st} + \frac{1}{s}e^{-st}\right]_{t=0}^\infty = \frac{1}{s^2} \quad s > 0$$

2. Compute the Laplace Transform of the function $f(t) = \sin 3t$, from the definition. (That means, do not use the Charts.)

   Solution Use formula (A.1.1):

   $$\mathcal{L}\{t\}(s) = \int_0^\infty \sin 3te^{-st}dt = \left[e^{-st}\frac{-s\sin 3t - 3\cos 3t}{s^2+9}\right]_{t=0}^\infty = \frac{3}{s^2+9}$$

3. Compute the Laplace Transform of the function $f(t) = \cos 2t$, from the definition. (That means, do not use the Charts.)

Solution Use formula (A.1.1):

$$\mathcal{L}\{t\}(s) = \int_0^\infty \cos 2te^{-st}dt = \left[e^{-st}\frac{2\sin 2t - s\cos 2t}{s^2+4}\right]_{t=0}^\infty = \frac{s}{s^2+4}$$

4. Compute the Laplace Transform of the function

$$f(t) = \begin{cases} \sin \pi t & if\ t \le 1 \\ 0 & if\ 1 < t \end{cases}$$

from the definition. (That means, do not use the Charts.)

Solution Use formula (A.1.1):

$$\mathcal{L}\{t\}(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^1 \sin \pi te^{-st}dt = \left[e^{-st}\frac{-s\sin \pi t - \pi\cos \pi t}{s^2+\pi^2}\right]_{t=0}^1$$

$$= e^{-s}\frac{\pi}{s^2+\pi^2} - \frac{-\pi}{s^2+\pi^2} = \frac{\pi(e^{-s}+1)}{s^2+\pi^2}$$

5. Compute the Laplace Transform of the function

$$f(t) = \begin{cases} \cos \pi t & if\ t \le 1 \\ -1 & if\ 1 < t \end{cases}$$

from the definition. (That means, do not use the Charts.)

Solution Use formula (A.1.1):

$$\mathcal{L}\{t\}(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^1 \cos \pi te^{-st}dt - \int_1^\infty e^{-st}dt$$

$$= \left[e^{-st}\frac{\pi\sin \pi t - s\cos \pi t}{s^2+\pi^2}\right]_{t=0}^1 - \left[\frac{e^{-st}}{-s}\right]_{t=1}^\infty$$

$$= e^{-s}\frac{s}{s^2+\pi^2} - \frac{-s}{s^2+\pi^2} - \frac{e^{-s}}{s} = \frac{s(e^{-s}+1)}{s^2+\pi^2} - \frac{e^{-s}}{s}$$

6. (Do not Submit This one.) Compute the Laplace Transform of the function $f(t) = t\sin \pi t$, from the definition. (That means, do not use the Charts.)

Solution Write $F(s) = \mathcal{L}\{t\}(s)$. So,

$$F(s) = \int_0^\infty t\sin \pi t e^{-st} dt = -\frac{1}{\pi}\int_0^\infty te^{-st} d(\cos \pi t)$$

$$= -\frac{1}{\pi}\left(\left[te^{-st}\cos \pi t \right]_{t=0}^\infty - \int_0^\infty \cos \pi t(e^{-st} - ste^{-st})dt \right)$$

$$= -\frac{1}{\pi}\left(0 - \int_0^\infty \cos \pi t e^{-st} dt + s\int_0^\infty t\cos \pi t e^{-st} dt \right)$$

Use Formula (A.1.1) (Assume $s > 0$):

$$= -\frac{1}{\pi}\left(-\left[e^{-st}\frac{\pi\sin \pi t - s\cos \pi t}{s^2+\pi^2} \right]_{t=0}^\infty + \frac{s}{\pi}\int_0^\infty te^{-st} d\sin \pi t \right)$$

$$= -\frac{1}{\pi}\left(-\frac{s}{s^2+\pi^2} + \frac{s}{\pi}\int_0^\infty te^{-st} d\sin \pi t \right) = \frac{s}{\pi(s^2+\pi^2)} - \frac{s}{\pi^2}\int_0^\infty te^{-st} d\sin \pi t$$

$$= \frac{s}{\pi(s^2+\pi^2)} - \frac{s}{\pi^2}\left(\left[te^{-st}\sin \pi t \right]_{t=0}^\infty - \int_0^\infty \sin \pi t(e^{-st} - ste^{-st})dt \right)$$

$$= \frac{s}{\pi(s^2+\pi^2)} - \frac{s}{\pi^2}\left(0 - \int_0^\infty \sin \pi t e^{-st} dt + s\int_0^\infty te^{-st}\sin \pi t dt \right)$$

Use Formula (A.1.1) again,

$$F(s) = -\frac{s}{\pi(s^2+\pi^2)} + \frac{s}{\pi^2}\left[e^{-st}\frac{-s\sin \pi t - \pi\cos \pi t}{s^2+\pi^2} \right]_0^\infty - \frac{s^2}{\pi^2}F(s)$$

$$= \frac{s}{\pi(s^2+\pi^2)} + \frac{s}{\pi(s^2+\pi^2)} - \frac{s^2}{\pi^2}F(s) \Longrightarrow$$

$$\frac{s^2+\pi^2}{\pi^2}F(s) = \frac{2s}{\pi(s^2+\pi^2)} \Longrightarrow F(s) = \frac{2s\pi}{(s^2+\pi^2)^2}$$

6.2 Solutions of Initial Value Problems

Use the Laplace Transform Charts available in the internet, to solve the problems in this section.

1. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 10y = 0, \quad \begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$$

   Solution. For the solution $y = \varphi(t)$ of the IVP , let $Y(s) = \mathcal{L}\{y\} = \mathcal{L}\{\varphi\}$. Apply $\mathcal{L}$ to the ODE:

$$(s^2Y - sy(0) - y'(0)) - 6(sY - y(0)) + 10Y = 0 \Longrightarrow$$

$$(s^2Y - 1) - 6sY + 10Y = 0 \Longrightarrow \quad Y = \frac{1}{s^2 - 6s + 10} \Longrightarrow$$

$$\mathcal{L}(y) = Y = \frac{1}{(s-3)^2 + 1} = \mathcal{L}\{e^{3t}\sin t\} \Longrightarrow y = e^{3t}\sin t$$

2. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0, \quad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

   Solution. For the solution $y = \varphi(t)$ of the IVP , let $Y(s) = \mathcal{L}\{y\} = \mathcal{L}\{\varphi\}$. Apply $\mathcal{L}$ to the ODE:

$$(s^2Y - sy(0) - y'(0)) + 6(sY - y(0)) + 13Y = 0 \Longrightarrow (s^2Y - s) + 6(sY - 1) + 13Y = 0 \Longrightarrow$$

$$Y = \frac{s+6}{s^2 + 6s + 13} = \frac{(s+3)}{(s+3)^2 + 4} + \frac{3}{(s+3)^2 + 4}$$

$$= \mathcal{L}\{e^{-3t}\cos 2t\} + \frac{3}{2}\mathcal{L}\{e^{-3t}\sin 2t\} \Longrightarrow$$

$$\mathcal{L}\{y\} = \mathcal{L}\left\{e^{-3t}\cos 2t + \frac{3}{2}e^{-3t}\sin 2t\right\} \Longrightarrow$$

$$y = e^{-3t}\cos 2t + \frac{3}{2}e^{-3t}\sin 2t$$

3. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2}+8\frac{dy}{dt}+16y=e^{-4t}, \qquad \begin{cases} y(0)=1 \\ y'(0)=0 \end{cases}$$

Solution. For the solution $y=\varphi(t)$ of the IVP , let $Y(s)=\mathcal{L}\{y\}=\mathcal{L}\{\varphi\}$. Apply $\mathcal{L}$ to the ODE:

$$(s^2Y-sy(0)-y'(0))+8(sY-y(0))+16Y=\mathcal{L}\{e^{-4t}\}=\frac{1}{s+4}\Longrightarrow$$

$$(s^2Y-s)+8(sY-1)+16Y=3\mathcal{L}\{e^{-t}\}=\frac{1}{s+4}\Longrightarrow$$

$$(s^2+8s+16)Y=\frac{1}{s+4}+s+8=\frac{1+s^2+4s+8s+32}{s+4}=\frac{s^2+12s+33}{s+4}\Longrightarrow$$

$$Y=\frac{s^2+12s+33}{(s+4)^3}=\frac{(s+4)^2+4s+17}{(s+4)^3}=\frac{(s+4)^2+4(s+4)+1}{(s+4)^3}\Longrightarrow$$

$$Y=\frac{1}{s+4}+4\frac{1}{(s+4)^2}+\frac{1}{(s+4)^3}=\mathcal{L}\{e^{-4t}\}+4\mathcal{L}\{te^{-4t}\}+\frac{1}{2}\mathcal{L}\{t^2e^{-4t}\}\Longrightarrow$$

$$\mathcal{L}\{y\}=Y=\mathcal{L}\left\{e^{-4t}+4te^{-4t}+\frac{1}{2}t^2e^{-4t}\right\}\Longrightarrow$$

$$y=e^{-4t}+4te^{-4t}+\frac{1}{2}t^2e^{-4t}$$

4. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+5y=\cos 2t, \qquad \begin{cases} y(0)=1 \\ y'(0)=0 \end{cases}$$

Solution. For the solution $y=\varphi(t)$ of the IVP , let $Y(s)=\mathcal{L}\{y\}=\mathcal{L}\{\varphi\}$. Apply $\mathcal{L}$ to the ODE:

$$(s^2Y-sy(0)-y'(0))-2(sY-y(0))+5Y=\mathcal{L}\{\cos 2t\}=\frac{s}{s^2+4}\Longrightarrow$$

$$(s^2Y-s)-2(sY-1)+5Y=\frac{s}{s^2+4}\Longrightarrow(s^2-2s+5)Y=\frac{s}{s^2+4}+s-2\Longrightarrow$$

$$Y = \frac{s + s^3 + 4s - 2s^2 - 8}{(s^2+4)(s^2-2s+5)} = \frac{s^3 - 2s^2 + 5s - 8}{(s^2+4)(s^2-2s+5)}$$
$$= \frac{as+b}{s^2+4} + \frac{cs+d}{s^2-2s+5}$$

We compute a, b, c, d:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 5 & -2 & 4 & 0 \\ 0 & 5 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ -8 \end{pmatrix} \longrightarrow$$

The inverse of this matrix (use TI-84 carefully)

$$A^{-1} = \frac{1}{17} \begin{pmatrix} -4 & -8 & 1 & 2 \\ 32 & -4 & -8 & 1 \\ 21 & 8 & -1 & -2 \\ -40 & 5 & 10 & 3 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{17} \\ -\frac{8}{17} \\ \frac{16}{17} \\ -\frac{24}{17} \end{pmatrix} \Longrightarrow$$

$$Y = \frac{1}{17}\frac{s-8}{s^2+4} + \frac{8}{17}\frac{2s-3}{s^2-2s+5} = \frac{1}{17}\frac{s-8}{s^2+4} + \frac{8}{17}\frac{2s-3}{(s+1)^2+4} \Longrightarrow$$

$$Y = \frac{1}{17}\frac{s}{s^2+4} - \frac{4}{17}\frac{2}{s^2+4} + \frac{8}{17}\frac{2(s+1)}{(s+1)^2+4} - \frac{20}{17}\frac{2}{(s+1)^2+4} \Longrightarrow$$

$$\mathcal{L}\{y\} = Y = \frac{1}{17}\mathcal{L}\{\cos 2t\} - \frac{4}{17}\mathcal{L}\{\sin 2t\} + \frac{16}{17}\mathcal{L}\{e^{-t}\cos 2t\} - \frac{20}{17}\mathcal{L}\{e^{-t}\sin 2t\} \Longrightarrow$$

$$y = \frac{1}{17}\cos 2t - \frac{4}{17}\{\sin 2t + \frac{16}{17}e^{-t}\cos 2t - \frac{20}{17}e^{-t}\sin 2t$$

6.3 Step Functions and Dirac Delta

1. Compute the Laplace Transform of the function

$$u_2(t) = \begin{cases} 0 & if\ t < 2 \\ 1 & if\ 2 \le t \end{cases}$$

2. Compute the Laplace Transform of the function

$$f(t) = \begin{cases} 0 & if\ t < 2 \\ 1 & if\ 2 \leq t < 3 \\ 0 & if\ 3 \leq t \end{cases}$$

3. Compute the Laplace Transform of the function

$$f(t) := d_{.01}(t-2) = \begin{cases} 0 & if\ t < 1.99 \\ 100 & if\ 1.99 \leq t < 2.01 \\ 0 & if\ 3 \leq t \end{cases}$$

   (It is the same function $d_\tau(t - t_0)$ in the notes.)

4. Compute the Laplace Transform of the Dirac Delta $\delta(t-2)$. (You can use the formula).

6.4 Systems with Discontinuous Functions

Some of the problems in § 6.2 could fall in this section.

1. Use Laplace Transform to solve the IVP,

$$\frac{d^2y}{dt^2} + 9y = u_\pi, \qquad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

   Solution. For the solution $y = \varphi(t)$ of the IVP , let $Y(s) = \mathcal{L}\{y\} = \mathcal{L}\{\varphi\}$. Apply $\mathcal{L}$ to the ODE:

$$(s^2Y - sy(0) - y'(0)) + 9Y = \frac{e^{-\pi s}}{s} \Longrightarrow (s^2Y - s) + 9Y = \frac{e^{-\pi s}}{s} \Longrightarrow$$

$$Y = e^{-\pi s}\frac{1}{s(s^2+9)} + \frac{s}{s^2+9} = e^{-\pi s}\frac{1}{9}\left(\frac{1}{s} - \frac{s}{s^2+9}\right) + \mathcal{L}\{\cos 3t\} \Longrightarrow$$

$$Y = e^{-\pi s}\frac{1}{9}\left(\mathcal{L}\{1\} - \mathcal{L}\{\cos 3t\}\right) + \mathcal{L}\{\cos 3t\}$$

$$= \frac{1}{9}\left(\mathcal{L}\{u_\pi(t)\} - \mathcal{L}\{\cos 3(t+\pi)\}\right) + \mathcal{L}\{\cos 3t\} = \frac{1}{9}\mathcal{L}\{u_\pi(t)\} + \frac{10}{9}\mathcal{L}\{\cos 3t\} \Longrightarrow$$

$$\mathcal{L}\{y\} = Y = \mathcal{L}\left\{\frac{1}{9}u_\pi(t) + \frac{10}{9}\mathcal{L}\{\cos 3t\}\right\} \Longrightarrow$$

$$y = \frac{1}{9}u_\pi(t) + \frac{10}{9}\mathcal{L}\{\cos 3t\}$$

Appendix A

Appendix

A.1 A Formula

Lemma A.1.1.

$$\int e^{\lambda t}\cos\mu t dt = e^{\lambda t}\frac{\mu\sin\mu t+\lambda\cos\mu t}{\lambda^2+\mu^2}$$

$$\int e^{\lambda t}\sin\mu t dt = e^{\lambda t}\frac{\lambda\sin\mu t-\mu\cos\mu t}{\lambda^2+\mu^2}$$

Proof.

$$I = \int e^{\lambda t}\cos\mu t dt = \frac{1}{\mu}\int e^{\lambda t} d\sin\mu t = \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t - \lambda\int \sin\mu t e^{\lambda t}dt\right)$$

$$= \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t + \frac{\lambda}{\mu}\int e^{\lambda t} d\cos\mu t\right)$$

$$= \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t + \frac{\lambda}{\mu}\left(e^{\lambda t}\cos\mu t - \lambda\int e^{\lambda t}\cos\mu t dt\right)\right)$$

$$= \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t + \frac{\lambda}{\mu}\left(e^{\lambda t}\cos\mu t - \lambda I\right)\right)$$

$$\left(\frac{\lambda^2+\mu^2}{\mu^2}\right) I = \frac{1}{\mu}\left(e^{\lambda t}\sin\mu t + \frac{\lambda}{\mu}\left(e^{\lambda t}\cos\mu t\right)\right) = e^{\lambda t}\frac{\mu\sin\mu t+\lambda\cos\mu t}{\mu^2}$$

So,

$$I = e^{\lambda t}\frac{\mu\sin\mu t+\lambda\cos\mu t}{\lambda^2+\mu^2}$$

Now,

$$J := \int e^{\lambda t}\sin\mu t dt = -\frac{1}{\mu}\int e^{\lambda t} d\cos\mu t = -\frac{1}{\mu}\left(e^{\lambda t}\cos\mu t - \lambda\int e^{\lambda t}\cos\mu t dt\right)$$

$$= -\frac{1}{\mu}\left(e^{\lambda t}\cos\mu t - \lambda e^{\lambda t}\frac{\mu\sin\mu t+\lambda\cos\mu t}{\lambda^2+\mu^2}\right)$$

$$= -\frac{1}{\mu}e^{\lambda t}\left(\cos\mu t - \frac{\lambda\mu\sin\mu t+\lambda^2\cos\mu t}{\lambda^2+\mu^2}\right)$$

$$= -\frac{1}{\mu}e^{\lambda t}\left(\frac{-\lambda\mu\sin\mu t+\mu^2\cos\mu t}{\lambda^2+\mu^2}\right) = e^{\lambda t}\frac{\lambda\sin\mu t-\mu\cos\mu t}{\lambda^2+\mu^2}$$