Chapter 6
§6.2 Solutions of Initial Value Problem

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Goals

- The Goal of this section is to use Laplace Transform to solve Initial value problems, second order linear equations (as in §3.1, 3.3, 3.4, 3.5, 3.6).
- This way, the methods may become more algebraic.
- Two theorem that follows would be instrumental for this method.

Theorem 6.2.1

Theorem 6.2.1: Suppose f is a continuous function on an interval $0 \leq t \leq A$.

- Assume f' is and is piecewise continuous on the interval $0 \leq t \leq A$.
- Assume there are constants K, a, M such that

$$|f(t)| \leq Ke^{at} \qquad \text{for all} \quad t \geq M$$

(*In words, f has (at most) exponential growth.*)

$$\text{Then,} \quad \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{1}$$

Corollary 6.2.2

By repeated application of Theorem 6.2.1 we have:

Corollary 6.2.2: Suppose f is a continuous function on an interval $0 \leq t \leq A$.

- Assume $f', f^{(2)}, \ldots, f^{(n-1)}$ are continuous, and $f^{(n)}$ is piecewise continuous on the interval $0 \leq t \leq A$.
- Assume there are constants K, a, M such that

$$|f^{(i)}(t)| \leq Ke^{at} \qquad \text{for all} \quad i = 0, 1, \ldots, n \quad \text{and} \quad t \geq M$$

Then, $\mathcal{L}\{f^{(n)}(t)\} =$

$$s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0) \qquad (2)$$

- The cases $n = 1$ (1) and $n = 2$ will be used more frequently:
  - $n = 1$ case:

$$\text{Then,} \quad \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{3}$$

  - $n = 2$ case:

$$\mathcal{L}\{f^{(2)}(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \tag{4}$$

1-1 Correspondence

The following allows us to use Laplace Transformation to solve DEs:

- Suppose f, g are two continuous functions on an interval $0 \leq t \leq A$. Then,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \quad \Longrightarrow \quad f = g \tag{5}$$

- If $\mathcal{L}\{f(t)\} = F(s)$, we write $\mathcal{L}^{-1}\{F(s)\} = f$, to be called the inverse Laplace transform of g.

- Solving: There is a chart of functions $f(t)$ and the Laplace transforms $\mathcal{L}\{f(t)\}$, in page 321.
- To solve initial value problems, when $y(0) = y_0$, $y'(0) = y'_0$ are given, we compute the Laplace transform $\mathcal{L}(\{\varphi(t)\}$ of the solution $y = \varphi(t)$ and use the chart to compare.

Sample I: Ex 5

Compute the Inverse Laplace Transform of $F(s) = \frac{2s+2}{s^2+2s+5}$.

- We have
$$F(s) = 2\frac{s+1}{(s+1)^2+2^2}$$
- By Formula 10 of the chart:
$$F(s) = 2\frac{s+1}{(s+1)^2+2^2} = 2\mathcal{L}\{e^{-t}\cos 2t\} = \mathcal{L}\{2e^{-t}\cos 2t\}$$
- So,
$$\mathcal{L}^{-1}\{F(s)\} = 2e^{-t}\cos 2t$$

Sample II: Ex 9

Compute the Inverse Laplace Transform of $F(s) = \frac{1-2s}{s^2+4s+5}$.

- We have

$$F(s) = \frac{1-2s}{s^2+4s+5} = \frac{1-2s}{(s+2)^2+1}$$

$$= 5\frac{1}{(s+2)^2+1} - 2\frac{s+2}{(s+2)^2+1}$$

- By Formula 9, 10:

$$F(s) = 5\mathcal{L}\{e^{-2t}\sin t\} - 2\mathcal{L}\{e^{-2t}\cos t\}$$

$$= \mathcal{L}\{5e^{-2t}\sin t - 2e^{-2t}\cos t\}$$

- So,

$$\mathcal{L}^{-1}\{F(s)\} = 5e^{-2t}\sin t - 2e^{-2t}\cos t$$

Sample III: Ex. 8 (edited)

Compute the Inverse Laplace Transform of

$$F(s) = \frac{5s^3 - 7s^2 - 4s}{(s^2 + 2s + 5)(s^2 - 2s + 2)}.$$

- To solve Ex 8, along with examples 1, 2, 3 the method of partial fractions was used. Review all these examples.

Solution

Use method of partial fractions:

$$F(s) = \frac{5s^3 - 7s^2 - 4s}{(s^2+2s+5)(s^2-2s+2)}.$$

$$= \frac{as+b}{s^2+2s+5} + \frac{cs+d}{s^2-2s+2}$$

$$= \frac{(as+b)(s^2-2s+2)+(cs+d)(s^2+2s+5)}{(s^2+2s+5)(s^2-2s+2)} =$$

$$\frac{s^3(a+c)+s^2(-2a+b+2c+d)+s(2a-2b+5c+2d)+(2b+}{(s^2+2s+5)(s^2-2s+2)}$$

- $$\begin{cases} a+c=5 \\ -2a+b+2c+d=-7 \\ 2a-2b+5c+2d=-4 \\ 2b+5d=0 \end{cases}$$
- In matrix form:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 0 & 2 & 0 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ -4 \\ 0 \end{pmatrix}$$

- Use TI84 (rref):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 0 \\ -2 \end{pmatrix}$$

- So,

$$a = 5, \quad b = 5 \quad c = 0 \quad d = -2$$

- So,

$$F(s) = \frac{5s+5}{s^2+2s+5} - \frac{2}{s^2-2s+2}$$

$$= 5\frac{(s+1)}{(s+1)^2+4} - 2\frac{1}{(s-1)^2+1}$$

- By Formula 10, 9:

$$F(s) = 5\mathcal{L}\{e^{-t}\cos 2t\} - 2\mathcal{L}\{e^{t}\sin t\}$$

$$F(s) = \mathcal{L}\{5e^{-t}\cos 2t - 2e^{t}\sin t\}$$

- So,

$$\mathcal{L}^{-1}\{F(s)\} = 5e^{-t}\cos 2t - 2e^{t}\sin t$$

Sample IV: Ex 12

Solve the IVP

$$y'' + 3y' + 2y = 0; \quad y(0) = 1, y'(0) = 0$$

- Let $y = \varphi(t)$ be the solution. Write $Y(s) = \mathcal{L}\{y\}$.
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y''+3y'+2y\} = \mathcal{L}\{0\} \Longrightarrow \mathcal{L}\{y''\}+3\mathcal{L}\{y'\}+2\mathcal{L}\{y\} = 0$$

- By (3), (4):

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = 0$$

$$[s^2Y(s) - s] + 3[sY(s) - 1] + 2Y(s) = 0$$

$$Y(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+2)(s+1)} = \frac{a}{s+2} + \frac{b}{s+1}$$

- So,

$$Y(s) = \frac{1}{s+2} + \frac{1}{s+1} = \mathcal{L}\{e^{-2t}\} + \mathcal{L}\{e^{-t}\} = \mathcal{L}\{e^{-2t} + e^{-t}\}$$

- So,

$$y = \mathcal{L}^{-1}\{Y(s)\} = e^{-2t} + e^{-t}$$

Sample V: Ex 14

Solve the IVP

$$y'' - 4y' + 4y = 0; \quad y(0) = 1, y'(0) = 1$$

- Let $y = \varphi(t)$ be the solution. Write $Y(s) = \mathcal{L}\{y\}$.
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{0\} \Longrightarrow \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$$

- By (3), (4):

$$[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 4Y(s) = 0$$

$$[s^2Y(s) - s - 1] - 4[sY(s) - 1] + 4Y(s) = 0$$

$$Y(s) = \frac{s-3}{s^2 - 4s + 4} = \frac{s-3}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

- So, formula 2 asn 11

$$Y(s) = \mathcal{L}\{e^{2t}\} + \mathcal{L}\{te^{2t}\} = \mathcal{L}\{e^{2t} + te^{2t}\}$$

- So,

$$y = \mathcal{L}^{-1}\{Y(s)\} = e^{2t} + te^{2t}$$

Sample VI: Ex 18

Solve the IVP

$$y^{(4)} - y = 0; \quad y(0) = 1, y'(0) = 0, y"(0) = 1, y^{(3)}(0) = 0$$

- Let $y = \varphi(t)$ be the solution. Write $Y(s) = \mathcal{L}\{y\}$.
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y^{(4)} - y\} = \mathcal{L}\{0\} \Longrightarrow \mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = 0$$

- By the theorem

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y^{(3)}(0)] - Y(s) = 0$$

$$[s^4Y(s) - s^3 - s] - Y(s) = 0$$

$$Y(s) = \frac{s^3+s}{s^4-1} = \frac{s}{s^2-1} = \frac{a}{s-1} + \frac{b}{s+1}$$

- So,

$$Y(s) = \frac{1}{2(s-1)} + \frac{1}{2(s+1)} = \frac{1}{2}\mathcal{L}\{e^t\} + \frac{1}{2}\mathcal{L}\{e^{-t}\}$$

$$= \mathcal{L}\{\frac{e^t+e^{-t}}{2}\}$$

- So,

$$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{e^t+e^{-t}}{2}$$

Sample VII: Ex 20 (edited)

Solve the IVP

$$y" + 9y = \cos 2t; \quad y(0) = 1, y'(0) = 0$$

- Let $y = \varphi(t)$ be the solution. Write $Y(s) = \mathcal{L}\{y\}$.
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y" + 9y\} = \mathcal{L}\{\cos 2t\} \Longrightarrow \mathcal{L}\{y"\} + 9\mathcal{L}\{y\} = \frac{s}{s^2+4}$$

- By the theorem

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{s}{s^2+4} \Longrightarrow$$

$$s^2Y(s) - s + 9Y(s) = \frac{s}{s^2+4}$$

$$Y(s) = \frac{s^3+5s}{(s^2+4)(s^2+9)}$$

- Use partial fraction: Write

$$\frac{s^3+5s}{(s^2+4)(s^2+9)} = \frac{as+b}{s^2+4} + \frac{cs+d}{s^2+9}$$

$$\begin{cases} a+c=1 \\ b+d=0 \\ 9a+4c=5 \\ 9b+4d=0 \end{cases}$$

- In matrix form:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 9 & 0 & 4 & 0 \\ 0 & 9 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \end{pmatrix}$$

- Use TI84 (rref):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} .2 \\ 0 \\ .8 \\ 0 \end{pmatrix}$$

- So, $a = .2, b = 0, c = .8, d = 0$ and

$$Y(s) = .2\frac{s}{s^2+4} + .8\frac{s}{s^2+9} = .2\mathcal{L}\{\cos 2t\} + .8\mathcal{L}\{\cos 3t\}$$

$$Y(s) = \mathcal{L}\{.2\cos 2t + .8\cos 3t\}$$

- So, the solution

$$y = \mathcal{L}^{-1}(Y(s)) = .2\cos 2t + .8\cos 3t$$

Sample VIII: Ex 26

Consider the IVP:

$$y'' + 4y = \begin{cases} t & if\ 0 \leq t < 1 \\ 1 & if\ 1 \leq t \leq \infty \end{cases} \qquad y(0) = 0, y'(0) = 0$$

Let $y = \varphi(t)$ be the solution. Compute $Y(s) = \mathcal{L}\{y\}$.

- Also, write $g(t) = \begin{cases} t & if\ 0 \leq t < 1 \\ 1 & if\ 1 \leq t \leq \infty \end{cases}$
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\} \Longrightarrow \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

- Formula 12, 13 could be used to compute $\mathcal{L}\{g(t)\}$. Compute directly:

$$\mathcal{L}\{g(t)\}(s) = \int_0^\infty e^{-st} g(t)dt = \int_0^1 e^{-st} tdt + \int_1^\infty e^{-st} dt$$

$$= \frac{1}{-s}\int_0^1 tde^{-st} + \left[\frac{e^{-st}}{-s}\right]_{t=1}^{\infty}$$

$$= \frac{1}{-s}\left[\left[te^{-st}\right]_{t=0}^{1} - \int_0^1 e^{-st} dt\right] + \frac{e^{-s}}{s}$$

$$= \frac{1}{-s}\left[e^{-s} + \left[\frac{e^{-st}}{s}\right]_{t=0}^{1}\right] + \frac{e^{-s}}{s}$$

$$= \frac{1}{-s}\left[e^{-s} + \left[\frac{e^{-s}}{s} - \frac{1}{s}\right]\right] + \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s^2}$$

- We have

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \Longrightarrow$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1-e^{-s}}{s^2}$$

- So,

$$Y(s) = \frac{1-e^{-s}}{s^2(s^2+4)}$$

§6.2 Assignments and Homework

- Read Example 2,3 (They are helpful).
- Homework: §6.2 See the Homework Site!