Math 221: Online Lecture Guidance

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1 Intrinsic definition of Determinant

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The definition of determinants by expansion by rows or columns are not so intrinsic, which they are very algorithmic. Question that follows such definitions is that why are all those row expansion or column expansion agree? Here I give a more intrinsic definition. Consider a square matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \text{of size } n \times n. \tag{1}$$

We consider products of n entries the type

$$a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

1. The subscripts indicate that a_{1i_1} comes from the 1^{st}-ow, a_{2i_2} comes from the n^{nd}-row, so on and a_{ni_n} comes from the n^{th}-row. So, we picked exactly one entry from each row, in this product $a_{1i_1} a_{2i_2} \cdots a_{ni_n}$.

2. In fact, $i_1, i_2, \ldots, i_n$ are chosen in such a way so that there is exactly one factor from each column in this product $a_{1i_1} a_{2i_2} \cdots a_{ni_n}$. This means, $i_1, i_2, \ldots, i_n$ is a rearrangement of the integers, $1, 2, \ldots, n$.

3. (**Definition:**) Such a rearrangement is called $i_1, i_2, \ldots, i_n$ is called a **permutation** of $1, 2, \ldots, n$. Permutations (and **combinations**) is an important topic in mathematics (and is sometimes popular among math majors and graduate students). So, a permutation is a 1-to-1 and onto map $\sigma : \{1, 2, \ldots, n\} \xrightarrow{\sim} \{1, 2, \ldots, n\}$.
   The above rearrangement $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$ corresponds to the $\sigma : \{1, 2, \ldots, n\} \xrightarrow{\sim} \{1, 2, \ldots, n\}$, given by $\sigma(k) = i_k$.

4. Given, a **permutation** σ, given by $i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, i_n$ of $1, 2, \ldots, n$, we get a new permutation, σ' by **switching** i_k, i_{k+1}. So, σ', given by $i_1, i_2, \ldots, i_{k+1}, i_k, \ldots, i_n$ of $1, 2, \ldots, n$. Such a "**switcharoo**" is called a **transposition**.

5. Now, you would agree that, given a permutation σ, given by $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$, we can apply a number of transpositions, to bring $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$ back to the natural order $\{1, 2, \ldots, n\}$. Here is a theorem:

   Theorem and Definitions. Suppose σ, is a permutation of $\{1, 2, \ldots, n\}$, given by $i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, i_n$.

   (a) The number of permutations needed to bring σ back to the natural order $1, 2, \ldots, n$ would vary, depending on how you do it.
   (b) Given σ, to bring it back to the natural order $1, 2, \ldots, n$, the number of transpositions needed, would either be **even** or **odd**.
   (c) (**Defintion**) Given σ,
      i. σ is said to be a **even permutation**, even number of transpositions are needed to bring it back to the natural order.
      ii. σ is said to be a **odd permutation**, odd number of transpositions are needed to bring it back to the natural order.
      iii. (**Defintion**) We define

$$\mathbf{Sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even permutation} \\ -1 & \text{if } \sigma \text{ is odd permutation} \end{cases}$$

Definition 1.1. Suppose $\mathbf{A}$ is a matrix as above (1). Define, determinant

$$\det(\mathbf{A}) = \sum_{\sigma} \mathbf{Sign}(\sigma) a_{1i_1} a_{2i_2} \cdots a_{ni_n} \qquad \text{where} \qquad \sigma(k) = i_k$$

and, the sum runs through all the permutations σ of $\{1, 2, \ldots, n\}$.

2 Comments on Derivatives of Matrices

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At the end § 5.2 a list of formulas were give on derivatives of matrices. Recall, and let

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix} \qquad \text{be a matrix}$$

1. Define derivative

$$\frac{d\mathbf{A}(t)}{dt} = \mathbf{A}'(t) = \left(\frac{da_{ij}(t)}{dt} \right)$$

2. Integral

$$\int \mathbf{A}(t)dt = \left(\int a_{ij}(t)dt \right), \quad \int_a^b \mathbf{A}(t)dt = \left(\int_a^b a_{ij}(t)dt \right)$$

3. It follows,

   (a) $\frac{d(c\mathbf{A})}{dt} = \frac{cd(\mathbf{A})}{dt}$ for any matrix c of constants.

   (b) $\frac{d(\mathbf{A}+\mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$

   (c) $\frac{d(\mathbf{AB})}{dt} = \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt}$

2.1 Comments on the Product Rule

Proof of the Product Rule Let $A(t) = (a_{ij}(t))$ be a matrix of size $m \times n$ and $B(t) = (b_{ij}(t))$ be a matrix of size $n \times p$. The product $A(t)B(t) = (c_{ij}(t))$ is a $m \times p$ matrix. We have

$$A(t)B(t) = (c_{ij}(t)) = \left(\sum_{k=1}^{n} a_{ik}(t)b_{kj}(t) \right)$$

So,

$$\frac{d(\mathbf{AB})}{dt} = \left(\frac{dc_{ij}(t)}{dt}\right) = \left(\sum_{k=1}^{n} \frac{d}{dt}(a_{ik}(t)b_{kj}(t))\right) = \left(\sum_{k=1}^{n} \frac{da_{ik}(t)}{dt}b_{kj}(t) + a_{ik}(t)\frac{db_{kj}(t)}{dt}\right)$$

$$= \left(\sum_{k=1}^{n} \frac{da_{ik}(t)}{dt}b_{kj}(t)\right) + \left(\sum_{k=1}^{n} a_{ik}(t)\frac{db_{kj}(t)}{dt}\right) = \frac{d\mathbf{A}(t)}{dt}\mathbf{B}(t) + \mathbf{A}(t)\frac{d\mathbf{B}(t)}{dt}$$

This completes the proof.

Here are few comments

1. For product rules for two functions, that would not be necessary. Like

$$\frac{d\left(f(t)g(t)\right)}{dt} = f'(t)g(t) + f(t)g'(t) = g(t)f'(t) + g'(t)f(t)$$

   We can use this to derive the power (chain) rule

$$\frac{f(t)^n}{dt} = nf(t)^{n-1}f'(t)$$

2. However, we need to be mindful of the order of multiplication in the product rule formula:

$$\frac{d(\mathbf{A}(t)\mathbf{B}(t))}{dt} \neq \mathbf{B}(t)\frac{d\mathbf{A}(t)}{dt} + \frac{d\mathbf{B}(t)}{dt}\mathbf{A}(t)$$

3. Likewise, power (chain) rule fails

$$\frac{d(\mathbf{A}^n(t))}{dt} \neq n\mathbf{A}^{n-1}(t)\frac{d\mathbf{A}(t)}{dt}$$

   For example, correctly we have,

$$\frac{d(\mathbf{A}^2(t))}{dt} = \frac{d\mathbf{A}(t)}{dt}\mathbf{A}(t) + \mathbf{A}(t)\frac{d\mathbf{A}(t)}{dt} \neq 2\mathbf{A}(t)\frac{d\mathbf{A}(t)}{dt}$$

2.2 Exponential of matrices

Most of what we expect seem to fail. While thinking of exponential, basic things thing to note:

1. The binomial expansion fails: For two square matrices A, B of order n,

   $(A+B)^2 = A^2 + AB + BA + B^2, \qquad \text{and} \quad (A+B)^2 \neq A^2 + 2AB + B^2$

Definition 2.1. Let $\mathbf{A} = (a_{ij})$ be a square matrix of size $n \times n$, with real entries. Most reasonable way to define exponential function

$$\exp(\mathbf{A}) := e^{\mathbf{A}} := \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n$$

(*Other possibility to define it as* $(e^{a_{ij}})$ *would be more simplistic!*)

1. Note, on the right hand side each entry is a infinite Series. One needs a proof that each entry converges. At this time, I do not know if it is true, I believe it is.

2. Also, $\exp(A+B) \neq \exp(A)\exp(B)$. Check this. So, the whole point is lost.

Derivatives Now suppose $\mathbf{A}(t) = (a_{ij}(t))$ whose entries are differentiable functions $a_{ij}(t)$.

1. Since the power (chain) rule fails:

   $$\frac{d\mathbf{A}^n(t)}{dt} \neq n\mathbf{A}^{n-1}(t)\frac{d\mathbf{A}(t)}{dt}$$

   the exponential derivative formula fails:

   $$\frac{d\exp(\mathbf{A}(t))}{dt} \neq \exp(\mathbf{A}(t))\frac{d\mathbf{A}(t)}{dt}$$

Remark: I started thinking about it because I wanted to define integrating factors for system of first order ODE, and solve them, as in Chapter 2. Method of Integrating factors do not work, in this case.