Diff. Equations (Hons)

Satya Mandal, University of Kansas, Lawrence, KS 66045

1 Symmetric Matrices (15 April 2020)

We write a symmetric matrix:

$$A = \begin{pmatrix} \mathbf{a_{11}} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n-1} & a_{1n} \\ a_{12} & \mathbf{a_{22}} & a_{23} & a_{24} & \cdots & a_{2n-1} & a_{2n} \\ a_{13} & a_{23} & \mathbf{a_{33}} & a_{34} & \cdots & a_{3n-1} & a_{3n} \\ a_{14} & a_{24} & a_{34} & \mathbf{a_{44}} & \cdots & a_{4n-1} & a_{4n} \\ a_{15} & a_{25} & a_{35} & a_{45} & \cdots & a_{5n-1} & a_{5n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n-1} & a_{2,n-1} & a_{3,n-1} & a_{4,n-1} & \cdots & \mathbf{a_{n-1,n-1}} & a_{n-1,n} \\ a_{1,n} & a_{2,n} & a_{3,n} & a_{4,n} & \cdots & a_{n-1,n} & \mathbf{a_{n,n}} \end{pmatrix}$$

Theorem 1.1. *Let A be a symmetric matrix. Then, there is an invertible matrix T such that $T^tAT = D$ is a diagonal matrix, where T^t denotes the transpose of T.*
(**Remark**: The way we defined "diagonalizeable" in the Diff. Equation Class is **not he same**.)

Proof. We use induction on n. We proceed in several steps:

1. If the first row (and hence first column) is zero, we use induction, by removing first row and first column. So, we assume first row has some non zero entry, $a_{1r} \neq 0$.
2. **Step I:** Goal is to reduce to the case to $a_{11} \neq 0$.
   We consider various cases:

(a) If $a_{11} \neq 0$, then there is nothing to do.

(b) Assume $a_{11} = 0$, but some diagonal entry $a_{kk} \neq 0$. In this case, for simplicity, assume $a_{22} \neq 0$.

$$T_1 = \begin{pmatrix} 0 & \mathbf{1} & 0 & \cdots & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{1} \end{pmatrix}$$

Then, $A_1 := T_1^t A T_1 = T_1 A T_1$ switches 1^{st} and 2^{nd} columns and then switches 1^{st} and 2^{nd} of A. The $(1,1)$-entry of A_1 is non zero.

(c) Finally, let all the diagonal entries $a_{kk} = 0$. However, we have $a_{1r} \neq 0$. For simplicity, assume $a_{12} \neq 0$. And, of course $a_{22} = 0$. Write

$$T_1 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{1} \end{pmatrix}$$

Now, $A_1 := T_1^t A T_1$, is obtained by adding 2^{nd} column A to its 1^{st} column and then adding 2^{nd} row, of the resulting matrix to its 1^{st} row. So, the $(1,1)$-entry of A_1 is $2a_{12} \neq 0$.

So, we found a matrix T_1, such that $A_1 := T_1^t A T_1$ has non zero $(1,1)$-entry. Note A_1 is symmetric.
Therefore, replacing A by A_1, we will assume $a_{11} \neq 0$.

3. **Step 2**. We have $a_{11} \neq 0$. We will subtract, suitable multiple of first column, from the other columns, and, then same for rows, to make all

the non diagonal entries of A zero. Write,

$$T_2 = \begin{pmatrix} \mathbf{1} & -\frac{\mathbf{a_{12}}}{\mathbf{a_{11}}} & -\frac{\mathbf{a_{13}}}{\mathbf{a_{11}}} & -\frac{\mathbf{a_{14}}}{\mathbf{a_{11}}} & \cdots & -\frac{\mathbf{a_{1n-1}}}{\mathbf{a_{11}}} & -\frac{\mathbf{a_{1n}}}{\mathbf{a_{11}}} \\ 0 & \mathbf{1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{1} \end{pmatrix}$$

Then,

$$T_2^t A T_2 = \begin{pmatrix} \mathbf{a_{11}} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{b_{22}} & b_{23} & b_{24} & \cdots & b_{2n-1} & b_{2n} \\ 0 & b_{23} & \mathbf{b_{33}} & b_{34} & \cdots & b_{3n-1} & b_{3n} \\ 0 & b_{24} & b_{34} & \mathbf{b_{44}} & \cdots & b_{4n-1} & b_{4n} \\ 0 & b_{25} & b_{35} & b_{45} & \cdots & b_{5n-1} & b_{5n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_{2,n-1} & b_{3,n-1} & b_{4,n-1} & \cdots & \mathbf{b_{n-1,n-1}} & b_{n-1,n} \\ 0 & b_{2,n} & b_{3,n} & b_{4,n} & \cdots & b_{n-1,n} & \mathbf{b_{n,n}} \end{pmatrix}$$

while we need not explicitly compute b_{ij}, the symmetry of the matrix $T_2^t A T_2$ follows, because $(T_2^t A T_2)^t = (T_2^t A^t T_2)^t = T_2^t A T_2$

Now, the proof is complete by induction. ∎