Vector Spaces
§4.5 Basis and Dimension

Satya Mandal, KU

October 23

Goals

Discuss two related important concepts:

- Define Basis of a Vectors Space V.
- Define Dimension $\dim(V)$ of a Vectors Space V.

Basis

Let V be a vector space (over $\mathbb{R}$). A set S of vectors in V is called a basis of V if

1. $V = Span(S)$ and
2. S is linearly independent.

- In words, *we say that S is a basis of V if S in linealry independent and if S spans V.*
- First note, it would need a proof (i.e. it is a theorem) that any vector space has a basis.

Continued

- The definition of basis does not require that S is a finite set.
  - However, we will only deal with situations when $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a finite set.
  - If V has a finite basis $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, then we say that V is finite dimensional. Otherwise, we say that V is infinite dimensional.

Examples from the Textbook

- **Reading Assignment:** §4.5 Example 1-5.
- **Example 1.** The set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of the 3−space $\mathbb{R}^3$.
  Proof.
  - Given any $(x,y,z) \in \mathbb{R}^3$ we have
    $$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$
    So, any $x,y,z) \in \mathbb{R}^3$ is a linearl combinations of elements in S. So, $\mathbb{R}^3 = Span(S)$.
  - Also, S us linealry independent:
    $$x(1,0,0)+y(0,1,0)+z(0,0,1) = (0,0,0) \implies x = y = z = 0.$$

Example 1a.

Similarly, a basis of the $n-$space $\mathbb{R}^n$ is given by the set

$$S = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$$

where

$$\mathbf{e_1} = (1, 0, \ldots, 0), \mathbf{e_2} = (0, 1, \ldots, 0), \ldots, \mathbf{e_n} = (0, 0, \ldots, 1).$$

This one is called the standard basis of $\mathbb{R}^n$.

Example 2 (edited)

The set $S = \{(1,-1,0),(1,1,0),(1,1,1)\}$ *is a basis of* $\mathbb{R}^3$.

Proof.

- First we prove $Span(S) = \mathbb{R}^3$. Let $(x,y,z) \in \mathbb{R}^3$. We need to find a, b, c such that

$$(x,y,z) = a(1,-1,0) + b(1,1,0) + c(1,1,1)$$

So,

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \textit{notationally} \quad A\mathbf{a} = \mathbf{v}$$

Continued

Compute inverse of A:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\textit{Add first row to second} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Subtract third row from first and subtract 2 times third row from second:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Continued

Multiply second row by .5; then subtract second row from first:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & .5 & .5 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & .5 & -.5 & 0 \\ 0 & 1 & 0 & .5 & .5 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

So,

$$A^{-1} = \begin{bmatrix} .5 & -.5 & 0 \\ .5 & .5 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Continued

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .5 & -.5 & 0 \\ .5 & .5 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence

$$(x, y, z) = a(1, -1, 0) + b(1, 1, 0) + c(1, 1, 1) \in Span(S).$$

Therefore, $Span(S) = \mathbb{R}^3$.

- Now, we prove S is linearly independent. Let
$$a(1,-1,0)+b(1,1,0)+c(1,1,1)=(0,0,0).$$
We need to prove $a=b=c=0$. In fact, in the matrix from, this equation is $A\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ where A is as above. Since, A is non-singular, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ So, S is linearly independent.
- Since, $span(S)=\mathbb{R}^3$ and S is linearly independent, S forms a bais of $\mathbb{R}^3$.

Examples 4

- Let P_3 be a vector space of all polynomials of degree less of equal to 3. Then $S\{1, x, x^2, x^3\}$ is a basis of P_3.
  Proof. Clearly $span(S) = P_3$. Also S is linearly independent, because

$$a1 + bx + cx^2 + dx^3 \quad \Longrightarrow a = b = c = d = 0.$$

■

Example 5.

- Let $\mathbb{M}_{3,2}$ be the vector space of all 3×2 matrices. Let

$$A_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{3,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, A_{3,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$A = \{A_{11}, A_{12}, A_{2,1}, A_{2,2}, A_{3,1}, A_{3,2}\}$$

is a basis of $\mathbb{M}_{3,2}$.

Theorem 4.9

Theorem 4.9(Uniqueness of basis representation): Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis of V. Then, any vector $\mathbf{v} \in V$ can be written in one and only one way as linear combination of vectors in S.

Proof. Suppose $\mathbf{v} \in V$. Since $Span(S) = V$

$$\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n} \quad where \quad a_i \in \mathbb{R}.$$

Now suppose there are two ways:

$$\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n} \quad and \quad \mathbf{v} = b_1\mathbf{v_1} + b_2\mathbf{v_2} + \cdots + b_n\mathbf{v_n}$$

We will prove $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

Subtracting $\mathbf{0} = (a_1 - b_1)\mathbf{v_1} + (a_2 - b_2)\mathbf{v_2} + \cdots + (a_n - b_n)\mathbf{v_n}$

Since, S is linearly independent,
$a_1 - b_1 = 0, a_2 - b_2 = 0, \ldots, a_n - b_n = 0$ or
$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$. The proof is complete. ■
Reading Assignment: §4.5 Example 6

Theorem 4.10

Theorem 4.10 (Bases and cardinalities) Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis of V, containing n vectors. Then any set containing more than n vectors in V is linearly dependent.

Proof. Let $T = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ be set of m vectors in V with $m > n$. For simplicity, assume $n = 3$ and $m = 4$. So, $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and $T = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$. To prove that T is dependent, we will have to find scalers x_1, x_2, x_3, x_4, not all zero, such that not all zero,

$$x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} + x_4\mathbf{u_4} = \mathbf{0} \qquad Equation - I$$

Subsequently, we will show that Equation-I has non-trivial solution.

Continued

Since S is a basis we can write

$$\begin{array}{rlll} \mathbf{u_1} = & c_{11}\mathbf{v_1} & +c_{12}\mathbf{v_2} & +c_{13}\mathbf{v_3} \\ \mathbf{u_2} = & c_{21}\mathbf{v_1} & +c_{22}\mathbf{v_2} & +c_{23}\mathbf{v_3} \\ \mathbf{u_3} = & c_{31}\mathbf{v_1} & +c_{32}\mathbf{v_2} & +c_{33}\mathbf{v_3} \\ \mathbf{u_4} = & c_{41}\mathbf{v_1} & +c_{42}\mathbf{v_2} & +c_{43}\mathbf{v_3} \end{array}$$

We substitute these in Equation-I and re-group:

$$\begin{array}{l} (c_{11}x_1 \quad +c_{21}x_2 \quad +c_{31}x_3 \quad +c_{41}x_4)\mathbf{v_1} \\ +(c_{12}x_1 \quad +c_{22}x_2 \quad +c_{32}x_3 \quad +c_{42}x_4)\mathbf{v_2} \\ +(c_{13}x_1 \quad +c_{23}x_2 \quad +c_{33}x_3 \quad +c_{43}x_4)\mathbf{v_3} \quad = \mathbf{0} \end{array}$$

Since $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent, the coeffents of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are zero. So, we have (in the next frame):

Continued

$$\begin{array}{lllll} c_{11}x_1 & +c_{21}x_2 & +c_{31}x_3 & +c_{41}x_4 & = 0 \\ c_{12}x_1 & +c_{22}x_2 & +c_{32}x_3 & +c_{42}x_4 & = 0 \\ c_{13}x_1 & +c_{23}x_2 & +c_{33}x_3 & +c_{43}x_4 & = 0 \end{array}$$

This is a system of three homogeneous linear equations in four variables. (less equations than number of variable. So, the system has non-tirvial (infinitley many) solutions. So, there are x_1, x_2, x_3, x_4, not all zero, so that Equation-I is valid. So, $T = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ is linealry dependent. The proof is complete. ■

Reading Assignment: §4.5 Example 7

Theorem 4.11

Suppose V is a vector space. If V has a basis with n elements then all bases have n elements.

Proof. Suppose $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ and $T = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ are two bases of V.

Since, the basis$S$ has n elements, and T is linealry independent, by the thoerem above m cannot be bigger than n. So, $m \leq n$.

By switching the roles of S and T, we have $n \leq m$. So, $m = n$. The proof is complete. ■

Reading Assignment: §4.5 Example 8.

Dimension of Vactor Spaces

Definition. Let V be a vector space. Suppose V has a basis $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ consisiting of n vectors. Then, we say n is the dimension of V and write $\dim(V) = n$. If V consists of the zero vector only, then the dimension of V is defined to be zero.

We have

- From above example $\dim(\mathbb{R}^n) = n$.
- From above example $\dim(P_3) = 4$. Similalry, $\dim(P_n) = n + 1$.
- From above example $\dim(\mathbb{M}_{3,2}) = 6$. Similarly, $\dim(\mathbb{M}_{n,m}) = mn$.

Dimensions of Subspaces

- If W is a subspace of V, one can prove, then

$$\dim(W) \leq \dim(V).$$

- **Reading Assignment:** §4.5 Example 9, 10, 11.

Dimensions of Subspaces: Examples

- **Example 9 (edited)**
  - $$Let \quad W = \{(x, y, 2x + 3y) : x, y \in \mathbb{R}\}$$

    The W is a subspace of $\mathbb{R}^3$ and $\dim(W) = 2$.
    Proof. Given $(x, y, 2x + 3y) \in W$, we have

    $$(x, y, 2x + 3y) = x(1, 0, 2) + y(0, 1, 3)$$

    This shows $span(\{(1, 0, 2), (0, 1, 3)\}) = W$. Also $\{(1, 0, 2), (0, 1, 3)\}$ is linearly independent. So, $\{(1, 0, 2), (0, 1, 3)\}$ is a basis of W and $\dim(W) = 2$. ■

Dimensions of Subspaces: Examples

Example 10 (edited) Let

$$S = \{(1, 3, -2, 13), (-1, 2, -3, 12), (2, 1, 1, 1)\}$$

and $W = span(S)$. Then W is a subspace of $\mathbb{R}^4$ and $\dim(W) = 2$.

- **Proof.** Denote the three vectors in S by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.
- Then $\mathbf{v_3} = \mathbf{v_1} - \mathbf{v_2}$. Write $T = \{\mathbf{v_1}, \mathbf{v_2}\}$.
- It follows, any linear combination of vectors in S is also a linear combination of vectors in T.
- $$So, \quad W = span(S) = span(T).$$
- Also T is linearly indpendent. So, T is a basis and $\dim(W) = 2$.

Theorem 4.12

Theorem 4.12. (Basis Tests): Let V be a vector space and $\dim(V) = n$.

- If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a linearly independent set in V (consisting of n vectors), then S is a basis of V.
- If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ span V, then S is a basis of V

Exercise

- **Exercise 12 (edited)**. Let $S = \{(4, -3), (12, -9)\}$. Why S is not a basis for $\mathbb{R}^2$?
  Answer: S is linearly dependent. This is immediate because the first vector is a multiple of the second.
- **Exercise 20 (edited)**. Why S is not a basis for $\mathbb{R}^3$ where

$$S = \{(6, 4, 1), (3, -5, 1), (8, 13, 6), (0, 6, 9)\}$$

  Answer: Here $\dim(\mathbb{R}^3) = 3$. So, any basis will have 3 vectors, while S has four.

Exercise

- **Exercise 23 (edited).** Let $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$. Why S is not a basis for P_2?
  Answer: $\dim P_2 = 3$ and S has 3 elements. So, we have to give different reason. In fact, S is linealry dependent:

$$3x^2 - 2x - 1 = 2(1 - x) - 3(1 - x^2)$$

Exercise

- **Exercise 28 (edited).** Why S is not a basis for $\mathbb{M}_{22}$, where

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Answer: $\dim(\mathbb{M}_{22}) = 4$ and S has 3 elements.

Exercise

Exercise 28 (edited). Is S forms a basis for $\mathbb{M}_{22}$, where

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Answer: $\dim(\mathbb{M}_{22}) = 4$ and S has 4 elements. Further, S is linearly independent. So, S is a basis of $\mathbb{M}_{22}$. To see they are linearly independent: Let

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + d \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a+b+c+d & c+d \\ b+d & a+b+c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = b = c = d = 0$$

Homework

Homework: §4.5 Exercise 7, 8, 9, 10, 15, 16, 17,, 18, 26, 27, 45, 51, 52, 56, 71, 72, 76.