Chapter 1: System of Linear Equations
§ 1.2 Gaussian Elimination

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Summer 2017: Fall 18 Update

Goals

We do the following in this section:

- Define of matrices.
- Define Elementary row operations on a matrices.
- Define matrices of the Row-echelon form.
- Elaborate Gaussian and Gauss-Jordan elimination.
- Solve systems of linear equations using Gaussian elimination (and Gauss-Jordan elimination).

Definitions

Defintion: For two positive integers m, n and $m \times n$-matrix is a rectangular array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Continued

- the array has m rows, horizontally placed, and it has n column, vertically placed.
- We say that the **size** of the above matrix is $m \times n$.
- A **square matrix** of order n is a matrix whose number of rows and columns are same and is equal to n.
- For a square matrix of order n, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **main diagonal entries**.

Continued

- Here a_{ij} is a real number, to be called $ij^{th}-$entry. This entry sits in the $i^{th}-$row $j^{th}-$column. The first subscript i of a_{ij} is called the row subscript and j is called the column subscript.
- It is possible to talk about matrices whose entries a_{ij} are not real numbers. We can talk about matrices of any kind of objects. However, in this course, we consider matrices with real entries ONLY, and such matrices are also called **real matrices**.
- We single out the matrices of complex numbers, whose entries are complex numbers.

The Augmented Matrix

Given a system of linear equations, we associate a matrix to be called the **augmented matrix** contains all the information regarding the system.
Consider the linear system of m equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases} \quad (1)$$

Continued

Definition: The **augmented matrix** of this system (1) is defined as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{pmatrix} \tag{2}$$

- Conversely, given a $m \times (n+1)$ matrix, we can write down a system of m linear equations in n unknowns (variables).

Continued

Definition: The **coefficient matrix** of this system (1) is defined as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}. \qquad (3)$$

Example 1.2.1

Consider the linear system (refer to the Example in § 1.1):

$$\begin{cases} x - 5y = 3 \\ -8x + 40y = 14 \end{cases}$$

Its augmented matrix of the system is

$$\begin{pmatrix} 1 & -5 & 3 \\ -8 & 40 & 14 \end{pmatrix}$$

and the coefficient matrix is

$$\begin{pmatrix} 1 & -5 \\ -8 & 40 \end{pmatrix}$$

Example 1.2.2

Consider the linear system (refer to the Example in § 1.1):

$$\begin{cases} 2x_1 + 4x_2 - x_3 = 7 \\ x_1 - 11x_2 + 4x_3 = 3 \\ 10x_1 - 6x_2 + 4x_3 = 3 \end{cases}$$

The augmented and the coefficient matrices of this system are:

$$\begin{pmatrix} 2 & 4 & -1 & 7 \\ 1 & -11 & 4 & 3 \\ 10 & -6 & 4 & 3 \end{pmatrix}; \qquad \begin{pmatrix} 2 & 4 & -1 \\ 1 & -11 & 4 \\ 10 & -6 & 4 \end{pmatrix}. \qquad (4)$$

Contnued

Recall, in § 1.1, for the system (4), an equivalent system in row-echelon form, was deduced:

$$\left\{\begin{array}{r} x_1 - 11x_2 + 4x_3 = 3 \\ x_2 - \frac{9}{26}x_3 = \frac{1}{26} \\ 0 = -31 \end{array}\right. \tag{5}$$

The augmented and coefficient of this equivalent system (5) are:

$$\begin{pmatrix} 1 & -11 & 4 & 3 \\ 0 & 1 & -\frac{9}{26} & \frac{1}{26} \\ 0 & 0 & 0 & -31 \end{pmatrix}; \qquad \begin{pmatrix} 1 & -11 & 4 \\ 0 & 1 & -\frac{9}{26} \\ 0 & 0 & 0 \end{pmatrix}$$

Example 1.2.3

Consider the linear system (refer to the Example in § 1.1):

$$\begin{cases} x_1 & & & +3x_4 & = 4 \\ & 6x_2 & -3x_3 & -3x_4 & = 0 \\ & 3x_2 & & -2x_4 & = 1 \\ 2x_1 & -x_2 & +4x_3 & & = 5 \end{cases} \tag{6}$$

Its augmented and the coefficient matrices are:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 6 & -3 & -3 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 2 & -1 & 4 & 0 & 5 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 6 & -3 & -3 \\ 0 & 3 & 0 & -2 \\ 2 & -1 & 4 & 0 \end{pmatrix}.$$

Continued

Recall that an equivalent system in row-echelon form, for the system (6), was deduced in § 1.2:

$$\begin{cases} x_1 & & & +3x_4 & = 4 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} \\ & & & x_4 & = 1 \end{cases}$$

The augmented and the coefficient matrices of this echelon form are given by:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -.5 & -.5 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Conclusion

The above discussions and examples demonstrate that the three basic operations that we used to reduce a lsystem (1) of linear equations to a row-echelon form, can be translated to a version for matrices.

Elementary Row operations

By an **elementary row operation** on a matrix we mean one of the following three:

- Interchange two rows.
- Multiply a row by a nonzero constant.
- Add a multiple of a row to another row.

Two matrices are said to be

$$\textbf{row} - \textbf{equivalent}$$

if one can be obtained from another by application of a sequence of elementary row operations.
Two row-equivalent matrices, correspond to two equivalent system of equations.

Row Echelon Form Matrices

Analogous to systems of linear equations in Echelon form, define:

Definition: A matrix is said to be in

$$\textbf{row} - \textbf{echelon form},$$

if it has the following properties:

- All rows consisting entirely of zeros occur at the bottom.
- First nonzero entry, in each non-zero row, is 1 (to be called the **leading 1**).
- For each successive nonzero rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

Continued

A matrix in row-echelon form is said to be

$$\textbf{in reduced row} - \textbf{echelon form},$$

if every column that has a leading 1 has zeros in every position above and below the leading 1.

Equivalent Row Echelon Matrix

Theorem: Suppose A is a matrix. Then, A is row-equivalent to a matrix B, which is in row-echelon form.
Proof. Similar to the proof of the analogous theorem for systems.

Method of Gaussian elimination

Consider a system of linear equations, as in (1).
A method of solving this system (1) is as follows:

- Write the augmented matrix of the system.
- Use the elementary row operations to reduce the augmented matrix to a matrix in row-echelon form.
- Write the linear system corresponding to the row-echelon matrix and solve by back-substitution.

This is known as the method of
Gaussian elimination with back-substitution,
in short by **Gaussian elimination**.

Example 1.2.4

Example: Use the method of Gaussian elimination to solve the system (6), using analogous steps. Recall the system:

$$\left\{\begin{array}{lllllr} x_1 & & & +3x_4 & = 4 & Eqn-1 \\ & 6x_2 & -3x_3 & -3x_4 & = 0 & Eqn-2 \\ & 3x_2 & & -2x_4 & = 1 & Eqn-3 \\ 2x_1 & -x_2 & +4x_3 & & = 5 & Eqn-4 \end{array}\right. \qquad (7)$$

Solution

The augmented matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 6 & -3 & -3 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 2 & -1 & 4 & 0 & 5 \end{pmatrix}.$$

We reduce this to row echelon form, by mirroring the reduction of the system (7) to echelon form.
Subtract 2 times row-1 from row-4:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 6 & -3 & -3 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & -1 & 4 & -6 & -3 \end{pmatrix}$$

Continued

$$\text{Multiply row 2 by } \frac{1}{6}: \begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & -1 & 4 & -6 & -3 \end{pmatrix}$$

Subtract 3 times row-2 from row-3 and add row-2 to Eqn-4:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1.5 & -.5 & 1 \\ 0 & 0 & 3.5 & -6.5 & -3 \end{pmatrix}$$

Continued

$$\text{Multiply row 3by } \frac{2}{3}: \quad \begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 3.5 & -6.5 & -3 \end{pmatrix}$$

Subtract 3.5 times row-3 from row-4:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & -\frac{16}{3} & -\frac{16}{3} \end{pmatrix}$$

Continued

Multiply row-4 by $\frac{3}{16}$:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad (8)$$

This matrix is in row-echelon form, and is row-equivalent to the augmented matrix.

Continued

The system of linear equations corresponding this row-echelon matrix (8) is

$$\begin{cases} x_1 & & & +3x_4 & = 4 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} \\ & & & x_4 & = 1 \end{cases}$$

By back-substitution:

$$x_4 = 1, \quad x_3 = \frac{2}{3} + \frac{1}{3} = 1, \quad x_2 = 1, \quad x_1 = 1.$$

Gauss-Jordan form

Definition. A matrix in row-echelon form is said to be in **Gauss-Jordan form**, if all the entries above leading entries are zero.

The method of Gaussian elimination with back substitution to solve system of linear equations can be refined by, first further reducing the augmented matrix to a Gauss-Jordan form and work with the system corresponding to it. This method is called **Gauss-Jordan elimination** method of solving linear systems.

Example 1.2.4a

Consider the system (7).
An equivalent matrix, in row-echelon form, is above (8):

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

All the entries above the leading 1 in row 2 is zero.

Continued

So, we try to achieve the same above the leading 1 in row 3. Add .5 times row 3 to row 2:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Continued

Now we want to get zeros above the leading 1 in row 4. Subtract 3 times the row 4 from row 1; add $\frac{2}{3}$ times the row 4 from row 2; add $\frac{1}{3}$ times the row 4 from row 3:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

This matrix is in Gauss-Jordan form.

Continued

The system of linear equation corresponding to this one is:

$$\begin{cases} x_1 & & & & = 1 \\ & x_2 & & & = 1 \\ & & x_3 & & = 1 \\ & & & x_4 & = 1 \end{cases}$$

So, the solution to the system is:

$$x_4 = 1, \qquad x_3 = 1, \qquad x_2 = 1, \qquad x_1 = 1.$$

Remark. If you feel comfortable working with matrices, it is best to reduce a system to Gauss-Jordan, instead of only to row-echelon form.

Example 1.2.5

Solve the following using Gaussian elimination or Gauss-Jordan elimination:

$$\begin{cases} x_1 & -\frac{x_2}{2} & +\frac{3x_3}{2} & = 12 \\ & 2x_2 & -x_3 & = 14 \\ 7x_1 & -5x_2 & & = 6 \end{cases}$$

The augmented matrix is

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{pmatrix}$$

Continued

Divide f second row by 2:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 7 & -5 & 0 & 6 \end{pmatrix}$$

Subtract 7 times first row from third row:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & -\frac{3}{2} & -\frac{21}{2} & -78 \end{pmatrix}$$

Continued

Add $\frac{3}{2}$ times second row to the third row:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & -\frac{45}{4} & -\frac{135}{2} \end{pmatrix}$$

Multiply third row by $-\frac{4}{45}$:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

This matrix is in row-echelon form.

Continued

So, we can use back substitution and solve the system. The system corresponding to this matrix is:

$$\begin{cases} x_1 & -\frac{1}{2}x_2 & +\frac{3}{2}x_3 & = 12 \\ & x_2 & -\frac{1}{2}x_3 & = 7 \\ & & x_3 & = 6 \end{cases}$$

By back-substitution:

$$x_3 = 6, \quad x_2 = 7 + \frac{1}{2}6 = 10, \quad x_1 = 12 - \frac{3}{2}6 + \frac{1}{2}10 = 8.$$

Alternate, Gauss-Jordan Method

Alternately, we could reduce the row-echelon matrix

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

to a Gauss-Jordan form. To do this add $\frac{1}{2}$ time the second row to the first:

$$\begin{pmatrix} 1 & 0 & 1.25 & 15.5 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

Continued

Subtract 1.25 times third rwo from the first:

$$\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

Now add .5 time the third row to the second:

$$\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

This matrix is in Gauss-Jordan form.

Continued

The system of linear equations corresponding to this matrix is:

$$\begin{cases} x_1 & & & = 8 \\ & x_2 & & = 10 \\ & & x_3 & = 6 \end{cases}$$

This gives the solution of our system.

Example 1.2.6

Solve the following using Gaussian or Gauss-Jordan elimination:

$$\begin{cases} 2x_1 & & +3x_3 & = 3 \\ 4x_1 & -3x_2 & +7x_3 & = 5 \\ 6x_1 & -9x_2 & +12x_3 & = 7 \end{cases}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 6 & -9 & 12 & 7 \end{pmatrix}$$

We will reduce this matrix to row-echelon form.

Continued

Subtract 2 times first row from second row and subtract 3 times first row from 3rd row:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{pmatrix}$$

Subtract 3 times the second row from third:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Continued

Divide first row by 2 and second row by -3:

$$\begin{pmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{which is in row echelon form.}$$

Continued

The system corresponding to this equation is:

$$\begin{cases} x_1 \qquad\quad + \frac{3}{2}x_3 = \frac{3}{2} \\ \qquad x_2 - \frac{1}{3}x_3 = \frac{1}{3} \\ \qquad\qquad\qquad 0 = 1 \end{cases}$$

The last equation is absurd. So, the system is inconsistent.

Example 1.2.7

Solve the following using Gaussian elimination or Gauss-Jordan elimination:

$$\left\{\begin{array}{rrrr} x & +2y & +z & = 8 \\ -4x & -8y & -4z & = -29 \end{array}\right.$$

The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 8 \\ -4 & -8 & -4 & -29 \end{pmatrix}$$

Continued

Add 4 times first row to the second row:

$$\begin{pmatrix} 1 & 2 & 1 & 8 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad \text{which is in row echelon form.}$$

The corresponding system of linear equations is

$$\begin{cases} x & +2y & +z & = 8 \\ & & 0 & = 3 \end{cases}$$

The last equation is absurd. So, the system is inconsistent.

Example 1.2.8

Solve the linear system corresponding to the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The matrix is already in row echelon form. The system is:

$$\begin{cases} x_1 & +x_2 & & = 1 \\ & x_2 & +x_3 & = 0 \end{cases}$$

$$\text{So,} \quad x_2 = -x_3, \quad x_1 = 1 - x_2 = 1 + x_3.$$

Continued

With $x_3 = t$, a parametric solution is

$$\begin{cases} x_1 = 1 + t \\ x_2 = -t \\ x_3 = t. \end{cases}$$

Example 1.2.9

Consider the system of linear equations.

$$\left\{\begin{array}{cccccc} x & +y & & = 0 & & Eqn-1 \\ & y & +z & = 0 & & Eqn-2 \\ x & & +z & = 0 & & Eqn-3 \\ ax & -by & +2cz & = 0 & & Eqn-4 \end{array}\right.$$

Find the values of a, b, c such that the system has (a) a unique solution, (b) no solution (c) an infinite number of solution.

Continued

Solution: The augmented matrix of the equation:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ a & -b & 2c & 0 \end{pmatrix}$$

Subtract 1 times first row from third and a times first row from fourth:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -b-a & 2c & 0 \end{pmatrix}$$

Continued

Add second row to third:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -b-a & 2c & 0 \end{pmatrix}$$

Divide third row by 2:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -b-a & 2c & 0 \end{pmatrix}$$

Continued

Add $-(a+b)$ second row to fourth:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2c-a-b & 0 \end{pmatrix}$$

Subtract $2c-a-b$ times third row from fourth:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Continued

The matrix is in row-echelon form. The corresponding liner system is:

$$\begin{cases} x & +y & & = 0 \\ & y & +z & = 0 \\ & & z & = 0 \\ & & 0 & = 0 \end{cases}$$

The system is consistent for all values of a, b, c, and by back substitution the system has unique solution $x = y = z = 0$.