Math 290: Online Guidance

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Contents

Chapter 1

System of Linear Equations

1.1 §1.1 Intro to Linear Systems

System of Linear Equations: In section, we essentially review what you already knew about solving linear equations. We also try to draw your attention that you may have been half aware.
You have read all of it, at least once.

Only § 1.2 has Homework from this chapter.

1. **slide 3:** , we define linear equations.

2. **slide 6:** , we define **system of linear equations**,
which means there are more that one equations.
The system is called **homogeneous**, if all the constant terms
$b_1 = 0, b_2 = 0, \cdots, b_m = 0$.

3. Examples 1.1.2 A, 1.1.2 B, 1.1.2 C demonstrate three possibilities.
A system may NOT have any solution (1.1.2 B).
Such a system is called **inconsistent system**
A system may have exactly one solution (1.1.2 A)
A system may have infinitely many solutions (1.1.2 C)
There is no other possibility. If a system has 2 solutions, then it must have infinitely many solutions.

4. **slide 11:** we state the above (3) as a **Theorem**.

5. **slide 12:** Two systems are defined to be **equivalent** if they have **same** set of solutions. *To solve linear equations you essentially, used three operations. This is how you create new* **equivalent** *systems, from the given system.*
   These three operations are listed in this slide.

6. **slide 13:** We define, when a system is said to be in **row echelon form**. *In fact, you solved linear systems in this manner, dropping variables, one-by one. If you were working is a system of 3 equations, in x, y, z, you retain one of the original 3 equation, then you cook-up a equation in y, z, then cook-up one equation in z only. This is how you obtain a system in* **row echelon form**, *equivalent to the given system.*
   The concept "row echelon form" is important.

7. **slide 15:** Theorem says any linear system can be reduced an equivalent system in **row echelon form**. *This process is given a name "Gaussian elimination". This is the process you always used to solve linear systems. In other words, we gave a name so something that you always knew.*

8. **Examples:** The rest of the section, we worked out examples, to reduce linear systems to **row echelon form**.

1.2 § 1.2 Gaussian elimination

Try to understand, that when you solved linear system of equations, the variable x, y, z or x_1, x_2, x_3, did not do anything. **You could as well, keep them in your head, and do the same.** That is want we would do in this section. Given, a system of m equations in n variables, you can drop the variable names, and the equality sign, you get m rows of $(n+1)$ numbers. Such an array of numbers is called a **matrix**. *Computers do not like store unnecessary information. So, when you want to store the information in a linear system, you would store this matrix.* The entries of a matrix can be real numbers, or complex numbers (or anything else). In this course, we consider matrices of real numbers only.

1. **slide 3:** We define a **matrix**. This is a fundamental concept, in this course. Subsequently, we define size of a matrix and square matrix.

2. **slide 6-8:** Given a system of m linear equations, in n variables, we associate two matrices, to be called **augmented matrix** and coefficient matrix. Subsequently, we work out some problems on this.

3. **slide 15:** Imitating the elementary operations with the linear equations in a system, we define three **elementary row operations**, on a given matrix. (*As if,* **with the variable names in our head***, we are performing the operations on the equations.*)
   Two matrices A and B are defined to be **row equivalent** of one can be obtained from the other, by a sequence of row operations.

4. **slide 16:** We define, when a matrix is said to have **row echelon form**. *This is similar to row echelon form of a system of equations, but you do not see the zero rows when you work with equations.* Subsequently, we define **reduced** row echelon form. **Read the both definitions**.

5. **slide 18:** The Theorem states that any matrix can be reduced to a matrix, by an application of few elementary row operations.

6. **slide 19:** We show how to solve linear equations, by reducing the augmented matrix to a row echelon matrix.

7. **Solve Examples:** We solve examples in the rest of the section.

1.3 Application of Linear Systems

This section is for motivational purpose. Read the section. **Non linear problems are sometimes too difficult to solve**, to be of much use. That is why **Linear modeling wins**! You may be able to find a perfect non linear model for some real life problem, but that would be impossible to solve. So, we go for linear models, even if it is not so perfect model.

Chapter 2

Matrices

In this chapter we study matrices. In other words, we learn to add, multiply them and do similar other things.

2.1 Operations with Matrices

We would define matrix addition, multiplication, and study their properties.

1. **slide 3:** We repeat the definition of matrices, from last chapter. The one given in this slide has m rows and n columns. So, it is said to have size $m \times n$. If number of rows and columns of a matrix A are same $(= n)$, then it is called a **square matrix, of order** n. If a matrix A has only one column then we call it a **column matrix**. A matrix is called a **row matrices**, if it has only one row.

2. **slide 7:** If two matrices A, B have same size, we define $A+B$, by entry wise addition. The sum is not defined, unless they have same size.

3. **slide 8:** A number c would be referred to as a **scalar**, in this course. Given such a scalar c and a matrix A, we define scalar multiplication cA. In next few slides, we work out a few problems on matrix addition and scalar multiplication.

4. **slide 11,12:** For matrix A of size $m \times p$, and matrix B of size $p \times n$, define the product AB. If number of columns do not match with number of rows of B, then the product AB is **not defined. Read the definition carefully**. In next few slides, we work out a few problems on matrix.

5. **slide 15:** Consider the system of equations as in § 1.2, slide 6. This has m equations and n variables $x_1, \dots, x_n$. We can write it in a **matrix form:**

$$A\mathbf{x} = \mathbf{b}, \qquad \text{It looks like } \textbf{one equation} \text{ now.}$$

   In next few slides, we write down systems in matrix form.

2.2 Properties of Matrices

An addition or a multiplication is what would be more generally know as a **binary operation**. A **binary operation** is something that combines two objects to obtain another objects (*of same type or not*). So, matrix addition, scalar multiplication, and matrix product are binary operations. When you work with real (or complex) numbers, **you are used to some properties, namely associativity, commutativity, distributivity**. Whenever you have a **new** binary operation, you need to ask and check whether these properties would be valid.

So, matrix addition, scalar multiplication and matrix product are binary operation. In this section, we study properties of associativity, commutativity, distributivity of these three binary operations.

1. **slide 3:** We state the validity of associativity, commutativity, distributivity of addition and scalar multiplication.

2. **slide 4:** Again, given a binary operation, we ask whether there an identity. As $0 + x = x$ for all real numbers x, we say 0 is the identity for addition. Similarly, since $1 \cdot x = x$ for all real numbers $x \neq 0$, we say 1 is the identity for multiplication.

   In this say that matrix addition has an identity (namely the zero matrix).

3. **slide 4**(Continued): Given a real number x, we have $x+(-x)=0$. So, we say $(-x)$ is the additive inverse of x. If $x \neq 0$, we have $x \cdot \frac{1}{x} = 1$. So, we say $\frac{1}{x}$ is the multiplicative inverse of x.
   Analogously, if a binary operation has an identity, we ask whether some element x would have an inverse, with respect to this operation?
   In our context:
   Given any matrix, A, it has an inverse $-A$, meaning $A+(-A)=O$.

4. **slide 5:** We assert that, for matrix multiplication Associativity and Distributivity properties would be valid.
   Let me comment the proof of associativity $(AB)C = A(BC)$ would be messy. From definition, each entry of the $(AB)C$ would be a double sum. Some may not feel comfortable to deal with such double sums.

5. **slide 6-8:** We assert that the square matrix I_n acts as an identity for matrix product (*as formulated in the slides*).

6. **slide 9-11:** We restate the theorem about solutions of linear systems, and give a proof, using matrix form of the equations.
   Proof given in class would be asked in exams. *We would have take-home exams. You would have to have enough familiarity to be able find the proof in the notes and reproduce.*

7. **slide 13-15:** We define transpose of a matrix, and give its properties.

8. **slide 16-18:** Two things:
   Commutativity fails for matrix product: $AB \neq AB$ (not always)
   (*You may be used to expecting this.*)
   Cancellation fails $CA = CB$ with $C \neq O$ does not mean $A = B$.
   We give examples.

9. **slide 19:** We solve some equations with matrices. I suggest you first solve it algebraically, then substitute the value of X.

10. **slide 25:** Given a polynomial $f(x)$ and a square matrix A we evaluate $f(A)$. Pay attention how we treat the constant term.

2.3 The Inverse of Matrices

The identity matrix I_n of order n has the identity-property for matrix-product, that $AI_n = I_nA = A$ for all square matrices A of order n.
Now that we realize that there is an identity for product, we investigate, for a given square matrix A, whether or when A has an inverse. Then, if it has one, how to compute it? That is what we do in this section.

1. **slide 3:** We define invertibility of a matrix A, and its inverse B.

2. **slide 4:** We state and prove that if A has an inverse, then it is unique. (*Proofs will be asked in the exam.*)

3. **slide 7:** We compute inverse, by solving linear systems.

4. **slide 7:** We compute inverse, reducing the augmented matrix $[AI_n]$ to $[I_nB]$, by elementary row operations.

5. **slide 19,20:** For a order two matrix A, we define $\det(A)$. Then, use it to give a formula for the inverse A^{-1}, of A.

6. **slide 21:** We give some formulas to compute inverses, with proofs. (*Proofs will be asked in the exam.*)

7. **slide 23:** We establish cancellation property of invertible matrices, with proof. (*Proofs will be asked in the exam.*)

8. **slide 25:** When A is invertible, the system of equations $A\mathbf{x} = \mathbf{b}$, has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

9. **Examples:** We work out some examples.

10. **Remark:** We gave a few formulas to manipulate inverses. However, given a matrix A, **only direct method** we gave to compute A^{-1} is as in slide 7 (4).

2.4 Elementary Matrices

In this section, we show how elementary row operations on a matrices A, translates to multiplication (from left) EA by a so called **elementary matrix** E.

1. **slide 3:** We show how to construct elementary matrices, from the identity matrix I_n.
   We do an elementary row operation on I_n, we obtain an elementary matrix E.

2. **slide 4: Theorem:** Suppose E is an elementary matrix. The E has an inverse.

   Proof. I usually do this on the board: Write

   $$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   (a) Suppose E is obtained from I_n, by switching i^{th} and j^{th} rows of I_n.
   Then, $E^{-1} = E$ itself.
   For example, if E is obtained by switching i^{st} and j^{rd} rows of I_3. Then,

   $$E = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad E^{-1} = E$$

   You can check this by multiplying $EE = I_3$.

   (b) Suppose E is obtained from I_n, by multiplying i^{th} row of I_n by $c \neq 0$.
   Then, E^{-1} is obtained by by multiplying i^{th} row of I_n by $\frac{1}{c}$.
   For example,
   if E is obtained by multiplying the 3^{rd} row of I_3 by π. Then,

   $$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} \quad \text{and} \quad E^{-1} = \frac{1}{\pi} I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\pi} \end{pmatrix}$$

   You can check this by multiplying these two matrices.

(c) Suppose E is obtained from I_n,
by adding λ-times i^{th} row of I_n to j^{th}-row.
Then, E is obtained from I_n,
by adding $(-\lambda)$-times i^{th} row of I_n to j^{th}-row.
For example, E is obtained from I_3,
by adding π-times 3^{rd} row of I_3 to the 1^{st}-row.

$$E = \begin{pmatrix} 1 & 0 & \pi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E^{-1} = \begin{pmatrix} 1 & 0 & -\pi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

You can check this by multiplying these two matrices.

3. Next few slides we examine elementary matrices.

4. **slide 8: Theorem 2.1** If B is obtained from A
by performing an elementary row operation on A, then $B = EA$,
where E is the elementary matrix obtained from I_n
by performing the same operation on I_n.

5. Next few slides we give a proof of the above in some cases, and do some eamples.

6. **slide 13:** We (re)define two matrices A and B of same size to be row equivalent, if

$$A = E_1 E_2 \cdots E_k B \quad \text{where } E_i \text{ are elementary.}$$

7. **slide 14: Theorem 2.2** A square matrix A is invertible if and only if A is product of elementary matrices:

$$A = E_1 E_2 \cdots E_k \quad \text{where } E_i \text{ are elementary.}$$

Proof. Before you read the proof, you need to know that there are two "theorems" to be proved as follows:

(a) If $A = E_1 E_2 \cdots E_k$ where E_i are elementary then A has an inverse.

(b) If A has an inverse then $A = E_1 E_2 \cdots E_k$ where E_i are elementary.

To prove (7a), with $B = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1}$, $AB = BA = I_n$.
Now, also read the proof of (7b).

8. **remaining slides:** We use this method to compute inverse of a square matrix.

9. **Column Operations:** Everything said in this section works if we replace the word "row" by **column**.
We can define two matrices A, B to be **column equivalent** on A is obtained by performing a few elementary column operations.
Theorem: A, B are column equivalent if and only if

$$A = BE_1E_2 \cdots E_k \quad \text{where } E_i \text{ are elementary.}$$

Chapter 3

Determinant

In this chapter we define determinant of a square matrix.

3.1 The Determinant of a Matrix

1. **slide 3:** For a square matrix A, of order n, we associates a number $\det(A)$, to be called the determinant of A.

   We explain that we define it inductively:
   First, define $\det(A)$, for square matrices A, of order 1.
   Uses the above, to define $\det(A)$, for square matrices A, of order 2.
   Uses the above, to define $\det(A)$, for square matrices A, of order 3
   Continue, until you reach order n.
   (*Such inductive process are very good for implementing in computer programming.*)

2. **slide 4:** We give a direct formula $\det(A)$ for square matrices A, of order 2.

3. **slide 7,8:** For a square matrices A, of order 3, we define the **minors** M_{ij} of A, for $i, j = 1, 2, 3$.

4. **slide 9:** By an adjustment of sign, we define **cofactor** $C_{ij} = (-1)^{i+j} M_{ij}$.

5. **slide 10:** For a square matrices A, of order 3, we define $\det(A)$ by **expanding by first row.**

6. **slide 20:** We give an alternate way to compute $\det(A)$, for square matrices A, of order 3.
   This method would not work for matrices of order 4 and above.

7. **slide 23:** For a square matrices A, of order n, we define the **minors** M_{ij} of A, for $i, j = 1, 2, \ldots, n$.

8. **slide 24:** By an adjustment of sign, we define **cofactor** $C_{ij} = (-1)^{i+j} M_{ij}$.
   We define $\det(A)$ by **expanding by first row.**

9. **slide 25: Theorem3.1.1:** To define $\det(A)$ **expansion by any row works.**
   In fact, expansion by any column works.

10. **slide 26, 27:** I give a more intrinsic definition of $\det(A)$.
    (*Skip it, if you want.*)

11. **slide 28:** Define triangular matrices.

12. **slide 29: Theorem3.1.2:** For a triangular matrix,
    $\det(A) =$ product of the diagonal entries.
    We give a proof. Do not skip the proof.

13. **The rest:** Problem solving.

3.2 $\det(A)$: using elementary operations

1. **slide 4: Theorem:** We give rules, how
   $\det(A)$ and $\det(B)$ would change/differ,
   if B is obtained from A by and elementary **row** operation.

2. **slide 5:** Same rules work for elementary **column** operation.

3. **The rest:** Problem solving.
   By application of a few elementary operation,
   we try to reduce the given matrix A to a triangular matrix T
   We keep track of changes in the determinant, in each step.
   And $\det(T)$ =product of the diagonal entries.

3.3 Properties of Determinants

1. **slide 4: Theorem 3.3.1** $\det(AB) = \det(A)\det(B)$.
   Try to understand the proof, as best as you can.

2. **slide 15: Theorem 3.3.4** $\det(A^{-1}) = \frac{1}{\det A}$ if A is invertible.
   Read this proof.

3. **slide 17: Theorem 3.3.5**

   $$A \quad \text{is invertible} \quad \iff \quad \det(A) \neq 0$$

   (I use the symbol $\iff$ to abbreviate "i*implies and implied by*")

4. **slide 20: Theorem 3.3.6 This is an important summary**, regarding invertible matrices.
   There are 5 equivalent conditions.
   Make sure you understand this slide.

5. **slide 22:** $\det(A) = \det(A^T)$

3.4 Applications of $\det(A)$

Main Application is defining $Adj(A)$
Giving formula for A^{-1}
Solving Linear Systems using Cramer's Rule.

1. **slide 4:** Given a square matrix, define the cofactor matrix $C = (C_{ij})$, where C_{ij} are the cofactors of A.

2. **slide 5:** Define adjoint $Adj(A) = C^T$

3. **slide 6: Theorem 3.4.1** Let A be square matrix of order n. Then,

$$A(Adj(A)) = (Adj(A))A = \det(A)I_n$$

   (*Try to understand the proof, as best as you can.*)
   Also, if $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} Adj(A)$$

4. **slide 17, 18: Cramer's Rule** to solve Linear Equtions.
   It is beautiful formula to solve Linear systems.

5. **slide 30:** A formula to find **area** of a triangle.

6. **slide 32:** A formula to find **volume** of a tetrahedron.

3.5 Rank of a matrix

We define rank of a matrix A, by the highest non vanishing minor.
Suppose A is a $m \times n$-matrix. For $1 \leq r \leq \min\{m, n\}$,

1. A square sub-matrix M of A, **of order** r is is obtained by removing $m - r$ rows and $n - r$ columns of A.
2. An r-minor is the determinant of such a sub-matrix M, **of order** r. (*We will not use this.*)
3. Define

$$rank(A) = \max\left\{r : \exists \text{ a square sub matrix } M \text{ of order } r, \quad \ni \quad \det(M) \neq 0\right\}$$

Exercise 3.5.1. Suppose A is a square matrix, of order n

1. If A is invertible (also called nonsingular), then $rank(A) = n$
2. If A is not invertible (also called singular), then $rank(A) < n$
3. If $A \neq 0$, then $rank(A) \geq 1$

Remark. In future we would define rank of matrix A and
of a Linear Transformation (to be defined).
That definitions would look more abstract.
But all that would be same as this definition.

Chapter 4

Vector Spaces

In this chapter, we start introducing materials that would be more formal or abstract. We would steer away from working with numbers, and start working with sets and other abstract objects. This may would appear as a new way to do mathematics, for many. Some of you would find it very interesting.

4.1 The n-space $\mathbb{R}^n$ and vectors in $\mathbb{R}^n$

1. We would use the concept **set**. By a **set** S, we mean a collection (a basket) of objects. Objects is S are also called elements (*and other things, more specific to the context*).

2. $\mathbb{R}$ will denote the set of all real numbers.
   ($\mathbb{C}$ *denotes the set of all complex numbers, which we may not discuss.*)

3. **slide 4:** We introduce the set $\mathbb{R}^n$,
   whose elements are n-tuples $(x_1, x_2, \ldots, x_n)$ of real numners.
   Elements $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ are also called **vectors**.

   (a) So, $\mathbb{R}^2$ denotes the set of ordered pairs (x, y) of real numbers.
   Elements (x, y) in $\mathbb{R}^2$ are in one-to-one correspondence to the points in the real plane.

   (b) So, $\mathbb{R}^3$ denotes the set of ordered pairs (x, y, z) of real numbers. Elements (x, y, z) in $\mathbb{R}^3$ are in one-to-one correspondence to the points in the 3 dimension space.

   (c) *However, there is no way to geometrically visualize* $\mathbb{R}^4$, $\mathbb{R}^5$ *and higher dimensional spaces.*

4. We will use the symbol "$\in$" to abbreviate the word "**in**".

5. **slide 5:** For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we define addition
   $\mathbf{u} + \mathbf{v}$, by coordinate wise addition.
   real numbers $c \in \mathbb{R}$, we define **scalar multiplication**
   $c\mathbf{u}$ by multiplying each coordinate by c.
   (*A real number c will also be referred to as* **scalar**.)

6. **slide 6,7:** We list 10 properties of
   vector addition and scalar multiplication.
   (*These are analogous to the properties of*
   matrix addition and scalar multiplication.)
   Most importantly, EXACTLY these 10 properties would be used
   to define **Vector Spaces** in the next section.

7. **slide 8:** Clearly, a vector $\mathbf{u} \in \mathbb{R}^n$ can be seen as row matrix.
   We also write it as a column matrix, depending on the context or convenience.

8. **slide 9:** *We talk about the concept of "vectors" in physics and mechanics. Other than borrowing of the word "vector" from physics, this will not be of any further help to us.*

9. **the rest:** Read the rest of the section.

4.2 Vector Spaces

We define Vector spaces in this section. The real n-space $\mathbb{R}^n$ under the addition and scalar multiplication would be the main examples of such vector spaces.

1. **slide 3, 4:** We define **binary operations**.
   (*We would not harp on it.*
   But conceptually it is basic.
   We are giving a name to something that you deal with regulary))

   You have been used to additions and multiplications, in many different context. Examples are additions and multiplication on $\mathbb{R}$, $\mathbb{C}$, addition, scalar multiplication and product of matrices, and in last section addition, scalar multiplication on $\mathbb{R}^n$. These are called **binary operations**.

   An Abstract Example: Let $\mathcal{W}(English)$ be the set of all formal words made of English alphabets. Give two words $U, V \in \mathcal{W}(English)$, define product $U \star V$ by concatenation. Then, $\star$ is a **binary operations** on $\mathcal{W}(English)$.

2. **slide 6- 8:** We define **Vector Spaces.**
   As indicated in §4.1 (slide 6, 7), it is re-listing
   of the 10 properties of $\mathbb{R}^n$, in a more abstract setting.

   Vector spaces are also called **Linear Spaces**.
   In fact, it is more appropriate that we call them Linear Spaces.
   The name "Vector spaces" we borrowed from physics, and the tradition continued.

3. **slide 9-14:** We give standard examples.
   Only non standard examples we give are Example 3, 4.
   Read them. Read the proof that they are in deed vector spaces.

4. **slide 15, 17:** We prove the zero vector is unique.
   (*You cannot have two of them.*)
   We do the same about "additive inverse" of a given vector $\mathbf{u}$.
   (*The proof may be asked in the exam.*)

5. **slide 18-22:** We prove important properties of scalar multiplication: $0\mathbf{v} = 0$, $c\mathbf{0} = 0$. (*The proof may be asked in the exam.*)
6. **the rest:** Examples

4.3 Subspaces of Vector spaces

Given any kind of mathematical "object" we define, we also "sub-objects" W of a given object V. It is unlikely, that you would be familiar to such abstract "objects". But you may be vaguely familiar with geometric spaces, and their subspaces; sets and subsets. In this section we define and consider subspaces W of given vector spaces V.

1. **slide 4:** Given a vector spaces V, we define its subspaces W. (*We should have called it sub-vector-spaces. But it is customary to call them "subspaces", because it is clear from the context.*)
2. **slide 5:** Good Examples of subspaces of $\mathbb{R}^4$.
3. *We use the notation $S \subseteq T$ to abbrevaite that S is a subset of T.*
4. **slide 6, 7: Theorem:** Suppose V is a vector space,
and $W \subseteq V$ be subset.
To check it W is a subspace or not, we only
need to **check 3 conditions** (not all 10).
(*The proof may be asked in the exam.*)
5. **slide 8:** Trivial Examples (meaning obvious examples of subspaces, for any abstract vector space V)
6. **slide 9, 10:** More natural examples.
7. Give a set $\mathcal{U}$ (*to be thought of a the "universe"*),
and two subsets $S \subseteq \mathcal{U}$, $S \subseteq \mathcal{U}$.
We define intersection $S \cap T = \{x \in \mathcal{U} : x \in S \text{ and } x \in T\}$.
(*So, $S \cap T$ consists of elements that are common to both S and T*)
8. **slide 11:** Theorem: Intersection of two subspaces is a subspace.

9. **the rest:** Examples

10. **slide 17:** This example has **special relevance** to us.
Solutions space of any homogeneous system is a subspace.
In other words, for a matrix A of size $m \times n$, the set

$$H = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} \quad \text{is a subspace of } \mathbb{R}^n$$

4.4 Spanning and Linear Independence

We define two important concept.
The word "spanning" is means "**generating**", in our context.

1. **slide 4:** Given $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, a linear combination of them is

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$$

2. **slide 5-14:** Examples on linear combinations.

3. **slide 15:** Given a set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ of vectors in V,
define $Span(S)$ as the set of all linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

4. **slide 16-18:** Theorem: $Span(S)$ is a subspace of V.
In fact it is the smallest subspace containing S.
(*The proof may be asked in the exam.*)

5. **slide 17-31:** Examples.
Most natural examples are in slides 19, 20.
They are the unit vectors on the co-ordinate axes, which spans $\mathbb{R}^n$.

6. **slide 32:** We define the 2^{nd}-**important** concept of linear independence.
We say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent, if

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n \Longrightarrow c_1 = c_2 = \cdots = c_n = 0$$

We use the symbol $\Longrightarrow$ to abbreviate "implies". Use the symbol $\Longleftrightarrow$ to abbreviate "implies and implied by", which is same as "if and only if" discussed above.

7. **slide 34-36: Theorem:** $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent
   $\Longleftrightarrow$
   one vector $\mathbf{u}_i$ is linear combination of the rest.
   (*The proof may be asked in the exam.*)

8. **the rest:** Examples.
   The **most natural examples**
   of linearly independent vectors are in slide 37, 38.
   They coincide with the natural spanning set of $\mathbb{R}^n$ in slides 19, 20.

4.5 Basis and Dimension

For a vector space V we define an important number $\dim V$.
Intuitively, you know ideas that $\dim \mathbb{R}^2 = 2$, $\dim \mathbb{R}^3 = 3$, $\dim \mathbb{R}^n = n$.
This should coincide with our definition.

1. **slide 4:** A subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$ is called a **basis** of V, if $V = Span(S)$, and S is Linearly Independent.

2. **slide 6, 7:** The most natural example of a **basis** of $\mathbb{R}^3$ is
   $\{\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_1 = (0,0,1)\}$.
   More generally, in slide 7, we give the natural example of a **basis** of $\mathbb{R}^n$.

3. **slide 8-12:** Other natural examples of **basis**. Note same vectors space V (or $\mathbb{R}^n$) can have many bases.

4. **slide 13: Theorem:** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V,
   any vector $\mathbf{v} \in V$ can be written, **only in way**,
   as linear combination $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. (*The proof may be asked in the exam.*)

5. **slide 15: Theorem:** Needed to prove (*next theorem*) that
   any two bases of V has same number of elements.
   (*Proof is long. Try to understand or get as much as you can from it.*)

6. **slide 19: Theorem:** Any two bases have same number of elements.

7. **slide 20:** Define $\dim V$.
   (*This is the most important definition/concept in the course*)

8. **slide 21:** Usual examples.

9. **slide 22: Corollary:** If $W \subseteq V$ is a subspace, then $\dim W \leq \dim V$.

10. **slide 23, 24:** More examples.

11. **slide 25: Theorem Basis Test**.
    (*Very helpful.*)

12. **slide 27, 28:** More corollaries. There is a **typo** in slide 28:

$$\dim\left(span(S)\right) \leq m$$

13. **slide 29-32:** Examples

14. **slide 33-36:** A nice method to find a basis of subspaces of $\mathbb{R}^n$.

15. **Remark:** We discuss theory for abstract vector spaces V. However, most of our examples of vector spaces would be subspaces $V \subseteq \mathbb{R}^n$.

4.6 Rank of a Matrix II

Let A be a $m \times n$-matrix. We discussed rank of of A in §3.5.
We (re)define the same in the context of subspaces of $\mathbb{R}^n$.
The the Null space $N(A)$ of A, is defined to be the set of solutions of $A\mathbf{x} = \mathbf{0}$.
This is a subspace of $\mathbb{R}^n$.

1. **slide 4-6:** We define the row space of A, and
   the column space of A

2. **slide 7: Theorem:**
   If A and B are row equivalent, then

   $$rowSpace(A) = rowSpace(B)$$

   Likewise, for column spaces.
   (*Proof is fairly strait forward. Try to understand*)

3. **slide 12: Theorem:** Assume A and B are row equivalent
   (Typo *in the notes; I did not write this hypothesis*).
   If B is in row echelon form, then non zero rows of B is a basis of $rowSpace(A)$.

4. **slide 13:Theorem:**

   $$\dim\left(rowSpace(A)\right) = \dim\left(columnSpace(A)\right)$$

   Very useful theorem. Proof of long.

5. **slide 21: Defintion:** We (re)define

   $$rank(A) = \dim\left(rowSpace(A)\right) = \dim\left(columnSpace(A)\right)$$

6. **slide 22:** In § 3.5, we defined $rank(A)$.
   That coincides with this new defitntion.
   It needs a proof that they are same.
   Proof may not be difficult, but we skip it.

7. **slide 23:** We define null space of A:

   $$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

   This is the solutions space of the homogeneous linear system.

8. **slide 24: Theorem:** The null space $\mathbf{N}(A)$ is a subspace of $\mathbb{R}^n$.

9. **slide 26: Defintion:**

$$Nullity(A) = \dim(\mathbf{N}(A))$$

10. **slide 27: Theorem:**

$$Nullity(A) + rank(A) = n \qquad \text{the number of columns of } A$$

11. **slide 30: Theorem:** Let $\mathbf{x}_p$ be a particular solution of
the Linear system $A\mathbf{x} = \mathbf{b}$.
Then, any solution $\mathbf{x}$ can be written as

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h \qquad \text{where} \qquad \mathbf{x}_h \in \mathbf{N}(A)$$

12. **slide 32:** For a square matrix A, a summary of equivalent conditions, when A is invertible.
This summary is important.

13. **slide 35-72:** Numerous examples.

Chapter 5

Eigenvalues and Eigenvectors

We discuss Eigenvalues and Eigenvectors, of square matrices. This is one of most important concepts in Linear Algebra, and this would have immediate applications in the Diff. Equation course.

5.1 Eigenvalues and Eigenvectors of matrices

Suppose A is a square matrix of size $n \times n$.

1. **slide 3:** We define **eigen value** $\lambda \in \mathbb{C}$ of a square matrix A, which can be a complex number.
   Given an eigen value, λ of A, we also define a $\mathbf{x} \in \mathbb{C}^n$, with $\mathbf{x} \neq \mathbf{0}$, is called an **eigen vector**, if

   $$A\mathbf{x} = \lambda\mathbf{x} \qquad \begin{cases} \text{which is same as} & (\lambda I - A)\mathbf{x} = \mathbf{0} \\ \text{which is same as} & \mathbf{x} \in N(\lambda I - A) \end{cases}$$

2. **slide 4: Theorem:**

   $$\lambda \in \mathbb{C} \quad \text{is an eigen value of } A \quad \Longleftrightarrow \quad |\lambda I - A| = 0$$

3. **slide 4-7:** The **eigen space** is a the null space $E(\lambda) = N(\lambda I - A)$. While we avoided complex vectors, more precisely we have

$$E(\lambda) = \begin{cases} \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} & if\ \lambda \in \mathbb{R} \quad \text{This is a subspace of } \mathbb{R}^n \\ \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\} & if\ \lambda \in \mathbb{C} \quad \text{This is a subspace of } \mathbb{C}^n \end{cases}$$

   Note $\mathbf{0} \in E(\lambda)$, but it is NOT an eigen vector.

4. **slide 8:** We define characteristic equation of A as

$$|\lambda I - A| = 0$$

   In fact, the left side is a polynomial

   $|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda^{n-1} + c_0$ called the characteristic polinomial.

5. **slide 10, 11:** Eigen values of a triangular matrix A
   are the diagonal entries of A.
   As you can see they can repeat.

6. **The rest:** Examples, t2 compute eigen values and eigen vectors.

5.2 Diagonalization

1. **slide 3:** Two square matrices A, B are defined to be **similar**, if there is an invertible matrix P such that $B = P^{-1}AP$.
   A square matrix A said to be diagonalizable, if there is an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix.

2. **slide 4:** Two similar matrices A, B have same eigen value.
   So, if A is diagonalizable, and $P^{-1}AP = D$ is a diagonal matrix,
   then the eigen values of A are the diagonal entries of D.

3. **slide 6:** A square matrix A is diagonalizable $\Longleftrightarrow$
   A has n linearly independent eigen vectors.

   The size of A in $n \times n$.
   Suppose $P^{-1}AP = D$ is a diagonal matrix.
   Write $P = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{pmatrix}$. Then, $\mathbf{p}_i$ is an eigen vector of A, corresponding the i^{th}-diagonal entry of D.

   We consider the converse.
   You may find r eigen values $\lambda_1, \lambda_2, \ldots, \lambda_r$.
   Pick a basis of $E(\lambda_1)$, Pick a basis of $E(\lambda_2)$, $\cdots$, Pick a basis of $E(\lambda_r)$
   When you put them together, they form a linearly independent set.
   Suppose you have n of them. **Line them up**, and form the matrix $P = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{pmatrix}$, using these eigen vectors. Then, $P^{-1}AP = D$ is a diagonal matrix.

4. **slide 13: Theorem.** If A has n distinct eigen values,
   then it is diagonalizable.
   But converse need not be true.

5. **the rest:** Examples on diagonalizability.
   Most of the problems are on 2×2 or 3×3.
   So, you would have 2 or 3 eigen values, **or less**.
   Steps are: 1) compute all the eigen values
   2) For each eigen value λ, find a basis of $E(\lambda)$.
   If you get n (which is 2, or 3 for your problems) of them
   line them up to form P.

Chapter 6

Inner Product Spaces

In the vector spaces $\mathbb{R}^2$ or $\mathbb{R}^3$, we have the concept of length of a vector $\mathbf{v}$ and and angle between two vectors $\mathbf{u}$, $\mathbf{v}$. Same extends to the the $\mathbb{R}^n$, which is do in §6.1. Then, we extend the same to abstract vector spaces.

6.1 Length and Dot Product in $\mathbb{R}^n$

In this section, we discuss the length and angles in $\mathbb{R}^n$. In fact, angles between two vectors $\mathbf{u}$, $\mathbf{v}$, is defined by defining dot-product.

1. **slide 3:** We recall, a vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is represented by arrows.

2. **slide 5:** (Re)define length of $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, by usual formula:

$$\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2} = \text{distance from } \mathbf{0} \text{ to } \mathbf{v}$$

From geometry, the angle θ between $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ and $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is given by the formula

$$\cos\theta = \frac{u_1 v_1 + u_2 v_2}{\| \mathbf{u} \| \| \mathbf{v} \|} \qquad \text{Any Question?!} \qquad (6.1)$$

3. **slide 6-10:** Imitating the above, we define the length of $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. Then we give usual properties of length and examples.

4. **slide 11** Define distance $d(\mathbf{u}, \mathbf{v})$, between $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, and $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$.
Then, we give three usual (expected) properties of distance (like **triangular inequality**)

   Remark: In higher level math, any space X with such a **distance** $d(u, v)$ would be called a **Metric Space**.
One can give a one semester course on **Metric Spaces**.
(*It is good to know some jargons, this is one.*)

5. **slide 12-18:** Exampes

6. **slide 19:** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, define **dot product** of two $\mathbf{u} \cdot \mathbf{v}$.

7. **slide 20**: Give properties of dot product.
In next section, these properties will listed, as the definition of the Inner product spaces, for abstract vector spaces V.
So, what is dot product in this section, will be called the inner product in next section.

8. **slide 21-23:** We state and prove Cauchy-Schwartz Inequality:
$$|\mathbf{u} \cdot \mathbf{v}| \leq \| \mathbf{u} \| \| \mathbf{v} \|$$
Note, without this inequality (6.1) (slide 5), the formula for angle would not make sense.
(*Same Cauchy-Schwartz Inequality will be stated for Inner product spaces. Proof would be line-for-line same as this proof. So, I may not give the proof; and ask you to prove that theorem in the exam.*)

9. **slide 24:** Define angle θ between $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
Define when $\mathbf{u}, \mathbf{v}$ are orthogonal.

10. **slide 25:** Prove triangular inequality.

11. **slide 26:** Prove Pythagorean.

12. the rest: Examples

6.2 Inner Product Spaces

This section is mostly imitation, of last section, generalising the dot product to Inner Product.

1. **slide 3:** Define a vector space V, to be an Inner Product space.
   The definition is re-listing
   the properties of dot-product in last section (slide 20).

2. **slide 4:** Give further properties of Inner Product.

3. **slide 5:** For a inner product space V, define, length, distance and angle,
   same way as in last section.

4. **slide 6:** State Cauchy-Schwartz Inequality,
   Triangle Inequality,
   Pythagorean
   (and define orthogonality).

5. **slide 7:** Define Orthogonal Projection $Proj_{\mathbf{v}}\mathbf{u}$ of $\mathbf{u}$ on $\mathbf{v}$.

6. **slide 8:** State and Prove orthogonal projection is actually orthogonal.

7. **the rest:** Example
   The examples slide 10, 11
   (and similar with continuous functions, and integration)
   are the only examples we discuss **that are unlike** $\mathbb{R}^n$.
   So, they are special for this course.

6.3 Orthonormal Bases

Given an inner product space $(V, \langle -, - \rangle)$, and a basis $\mathbf{v_1}, \ldots, \mathbf{v}_n$, there is nice (algorithmic) way to construct a basis $\mathbf{u_1}, \ldots, \mathbf{u}_n$, such that

1. $\mathbf{u}_i \perp \mathbf{u}_j$ for all $i \neq j$ (orthogonal is denoted by $\perp$)
2. Length $\| \mathbf{u}_i \| = 1$ for all i

Such a basis is called orthonormal basis.
Most instructors do not include this section
I did it because I find it beautiful.
There is no homework in this section.
Read the section as best as you can.

Chapter 7

Linear Transformations

In formal mathematics, we study sets with some structure.
A vector space V is a set, together with some structure.
Others kinds of such structures you may know are spaces
(Like circles, spheres, cubes, and other sub spaces $X \subseteq \mathbb{R}^n$).
Given two such structures, V, W, we also study function $f : V \longrightarrow W$, that **preserve the structure**.
For examples, given two spaces $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$
the functions $f : X \longrightarrow Y$ that preserve the structure, are the continuous functions.

In this chapter, we study functions
$f : V \longrightarrow W$ from a vector space V to another vector space W
that would **preserve the vectors space structures**.

7.1 Definitions and Introduction

1. **slide 1-9** We give some overview, and some of set theoretic background, regrading functions. Glance through them. These slides are not in included for exams.

2. **slide 11** For two vector spaces V, W, we define
   Linear transformation $T : V \longrightarrow W$.

They are also called **Linear maps** or **Homomorphims**.
Conceptually, they are the set theoretic functions, that **preserves the vector spaces structure**.

3. **slide 12** We demonstrate that projection maps
$\mathbb{R}^3 \longrightarrow \mathbb{R}$ or
$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ are Linear transformations.

4. **slide 15** We demonstrate that we can
use **homogeneous polynomials**
to define Linear transformations:
$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

5. **slide 18 Important Example:**
We demonstrate that a matrix A of size $n \times n$, defines a Linear transformations:
$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \textbf{by} \quad T(\mathbf{x}) = A\mathbf{x}$$

6. **slide 20-23** More examples

7. **slide 24-28** We give non examples. We demonstrate, the non homogeneous polynomials do not work.

8. **slide 29** Projections (as in last chapter) we Linear transformations.