Math 290: Homework and Problems

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Contents

Chapter 1

System of Linear Equations

1.1 §1.2 Introduction

No Homework

1.2 Gauss Elimination

1. Consider the system of linear equations:

$$\begin{cases} x_1 & & +3x_3 & = -2 \\ 2x_1 & +x_2 & +x_3 & = 7 \\ -x_1 & +x_2 & +3x_3 & = -1 \end{cases}$$

   (a) Write down the augmented matrix.

   (b) Reduce the augmented matrix to a row echelon form.

   (c) Use Gauss Elimination method or Gauss Jordan elimination to solve the this system. If the system is inconsistent, say so.

2. Consider the system of linear equations:

$$\begin{cases} 2x_1 & -3x_2 & +4x_3 & = 10 \\ & 2x_2 & -x_3 & = 14 \\ 7x_1 & -3x_2 & -x_3 & = 20 \end{cases}$$

(a) Write down the augmented matrix.

(b) Reduce the augmented matrix to a row echelon form.

(c) Use Gauss Elimination method or Gauss Jordan elimination to solve the this system. If the system is inconsistent, say so.

3. Consider the system of linear equations:

$$\begin{cases} 2x_1 & -3x_2 & +4x_3 & = 2 \\ 12x_1 & -12x_2 & +22x_3 & = 15 \\ 10x_1 & -9x_2 & +18x_3 & = 13 \end{cases}$$

(a) Write down the augmented matrix.

(b) Reduce the augmented matrix to a row echelon form.

(c) Use Gauss Elimination method or Gauss Jordan elimination to solve the this system. If the system is inconsistent, say so.

4. Consider the system of linear equations:

$$\begin{cases} & x_2 & -3x_3 & = 2 \\ x_1 & & -2x_3 & = 1 \\ 3x_1 & -x_2 & -3x_3 & = 1 \end{cases}$$

(a) Write down the augmented matrix.

(b) Reduce the augmented matrix to a row echelon form.

(c) Use Gauss Elimination method or Gauss Jordan elimination to solve the this system. If the system is inconsistent, say so.

5. Consider the system of linear equations:

$$\begin{cases} x_1 & -x_2 & -3x_3 & = 2 \\ -3x_1 & +3x_2 & +9x_3 & = 2 \end{cases}$$

(a) Write down the augmented matrix.

(b) Reduce the augmented matrix to a row echelon form.

(c) Use Gauss Elimination method or Gauss Jordan elimination to solve the this system. If the system is inconsistent, say so.

6. Consider the system of linear equations:

$$\begin{cases} x_1 & -x_2 & -3x_3 & = 2 \\ -3x_1 & +3x_2 & +9x_3 & = -6 \end{cases}$$

   (a) Write down the augmented matrix.
   (b) Reduce the augmented matrix to a row echelon form.
   (c) Use Gauss Elimination method or Gauss Jordan elimination to solve the this system. If the system is inconsistent, say so.

7. Consider the system of linear equations:

$$\begin{cases} x_1 & +x_2 & -4x_3 & = 2 \\ 2x_1 & +2x_2 & -8x_3 & = 4 \\ x_1 & +4x_2 & -16x_3 & = 8 \end{cases}$$

   (a) Write down the augmented matrix.
   (b) Reduce the augmented matrix to a row echelon form.
   (c) Use Gauss Elimination method or Gauss Jordan elimination to solve the this system. If the system is inconsistent, say so.

8. Consider the system of linear equations:

$$\begin{cases} -x_1 & -3x_2 & -x_3 & +x_4 = -7 \\ x_1 & -4x_2 & -3x_3 & -4x_4 = -3 \\ x_1 & +5x_2 & +2x_3 & +6x_4 = -3 \\ 10x_1 & +4x_2 & -2x_3 & -2x_4 = 6 \end{cases}$$

   (a) Write down the augmented matrix.
   (b) Reduce the augmented matrix to a row echelon form.
   (c) Use Gauss Elimination method or Gauss Jordan elimination to solve the this system. If the system is inconsistent, say so.

Chapter 2

Matrices

2.1 Operations on Matrices

Homework Problems:

1. **On Addition and Scalar Multiplication**

   (a) Consider the matrices:

   $$A = \begin{pmatrix} 7 & 1 \\ .5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

   Compute the following. If not defined, say so:
   (1) $2A + B$, (2) $2A - 2B$, (3) $\pi A + B$.

   (b) Consider the matrices:

   $$A = \begin{pmatrix} 7 & 1 & 0 \\ -3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 7 & \pi \end{pmatrix}$$

   Compute the following. If not defined, say so
   (1) $2A + B$, (2) $2A - 2B$, (3) $\pi A + B$.

(c) Consider the matrices:

$$A = \begin{pmatrix} 7 \\ -3 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 0 \\ \pi \end{pmatrix}$$

Compute the following. If not defined, say so
(1) $2A + B$, (2) $2A - 2B$, (3) $\pi A + B$.

(d) Consider the matrices:

$$A = \begin{pmatrix} 7 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 7 & \pi \end{pmatrix}$$

Compute the following. If not defined, say so
(1) $2A + B$, (2) $2A - 2B$, (3) $\pi A + B$.

(e) Consider the matrices:

$$A = \begin{pmatrix} 7 & 1 \\ .5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 7 & \pi \end{pmatrix}$$

Compute the following. If not defined, say so:
(1) $2A + B$, (2) $2A - 2B$, (3) $\pi A + B$.

(f) Consider the matrices:

$$A = \begin{pmatrix} 7 & 1 \\ .5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Compute the following. If not defined, say so:
(1) $2A + B$, (2) $2A - 2B$, (3) $\pi A + B$.

2. **On Matrix Multiplication**

(a) Consider the matrices:

$$A = \begin{pmatrix} 7 & 1 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

If defined, compute AB, BA. If not defined, say so.

(b) Consider the matrices:

$$A = \begin{pmatrix} 7 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

If defined, compute AB, BA. If not defined, say so.

(c) Consider the matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix}$$

If defined, compute AB, BA. If not defined, say so.

(d) Consider the matrices:

$$A = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix}$$

If defined, compute AB, BA. If not defined, say so.

(e) Consider the matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix}$$

If defined, compute AB, BA. If not defined, say so.

(f) Consider the matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & b & c \\ 0 & v & w \\ 0 & 0 & z \end{pmatrix}$$

If defined, compute AB, BA. If not defined, say so.

(g) Consider the matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

If defined, compute AB, BA. If not defined, say so.

(h) Consider the matrix:

$$A = \begin{pmatrix} 7 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

If defined, compute A^2, A^3.

(i) Consider the matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix}$$

If defined, compute AB, BA.

(j) Consider the matrices:

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix}$$

If not defined, say so.

3. **Matrix Equations**

(a) Solve the following matrix equation:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hint: Multiply out the lefthand side, and equate two sided, entry wise. You will get four equations, in x, y, z, w. Solve these four equations.

(b) Solve the following matrix equation:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hint: Same as the above.

2.2 Properties of Matrix Operations

Homework Problems:

1. **Soving Equations in Matrix**

   (a) Suppose A, B are two known matrices, and X is an unknown matrix. Solve the following equations (*in each case, assume the respective matrix operations are defined*):

   i. $2X = 3A + B$

   ii. $3X + A = -B$

   iii. $3X + 4A = -B$

   iv. Assume AB is defined, and $3X + AB = -B$. Further, if

   $$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

   Then, compute X.

   (b) **On Algebra of Matrix Multiplication**
   Let A, B, C be matrices. (*In each case, assume the respective matrix operations are defined.*)

   i. Simplify: $(A + 2B)C$

   ii. Simplify: $(A + 2I_n)C$
   (*here A, C are square marines of order n and I_n is the identity matrix*).

   iii. Simplify: $(A + 2\mathbf{O})C$
   (*here A, C have size $m \times n$ and* $\mathbf{O}$ *is the zero matrix*).

   (c) Let

   $$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

   i. Compute the transpose A^T, B^T, C^T.

   ii. Compute $A(BC)$

   iii. Compute $C(AB)$

iv. Compute $C^T B^T A^T$. (Hint: Use (1(c)ii).

v. Compute $B^T A^T C^T$. (Hint: Use (1(c)iii).

2. **Polynomial Evaluation**

(a) Let $f(x) = x^2 - 2x + 1$. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \quad \text{Compute} \quad f(A).$$

(b) Let $f(x) = x^3 - 3x^2 + 3x + 1$. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \quad \text{Compute} \quad f(A).$$

(c) Let $f(x) = x^3 - 3x^2 + 3x + 1$. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{Compute} \quad f(A).$$

(d) Let $f(x) = x^2 - x + 1$. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \quad \text{Compute} \quad f(A).$$

(e) Let $f(x) = x^2 - x + 1$. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{Compute} \quad f(A).$$

2.3 Inverse of Matrices

1. **On Inverting Matrices, using Gauss-Jordan**

(a) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

(b) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

(c) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

(d) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

(f) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & -3 \\ 0 & 1 & 1 \end{pmatrix}$$

(g) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

(h) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

(i) Consider the following matrix A. If the inverse of A exists, compute A^{-1}, else say so.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. **Algebra of Inverting Matrices**

(a) Suppose A, B are two matrices, with

$$A^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Compute $(AB)^{-1}$, $(A^T)^{-1}$ and $((AB)^T)^{-1}$.

(b) Suppose A, B are two matrices, with

$$A^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}$$

Compute $(AB)^{-1}$, $(A^T)^{-1}$ and $((AB)^T)^{-1}$.

(c) Suppose A, B are two matrices, with

$$A^{-1} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Compute $(AB)^{-1}$, $(A^T)^{-1}$ and $((AB)^T)^{-1}$.

(d) Suppose A, B are two matrices, with

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Compute $(AB)^{-1}$, $(A^T)^{-1}$ and $((AB)^T)^{-1}$.

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Compute $(AB)^{-1}$, $(A^T)^{-1}$ and $((AB)^T)^{-1}$.

3. **On Solving (nonsingular) systems** Definition. A linear system $A\mathbf{x} = \mathbf{b}$ is said to be a nonsingular system, if the coefficients matrix is invertible.

(a) Solve the following nonsingular system of equations

$$\begin{cases} x & +2y & = 1 \\ x & +3y & = -1 \end{cases}$$

Hint: Use (1a).

(b) Solve the following nonsingular system of equations

$$\begin{cases} x_1 & +2x_2 & -2x_3 & = 1 \\ 2x_1 & +4x_2 & -3x_3 & = -1 \\ & x_2 & +x_3 & = 2 \end{cases}$$

Hint: Use (1f).

(c) Solve the following nonsingular system of equations

$$\begin{cases} x_1 & +x_2 & +x_3 & = 1 \\ x_1 & +2x_2 & +2x_3 & = -1 \\ x_1 & +x_2 & +2x_3 & = 2 \end{cases}$$

Hint: Use (1g).

(d) Let A be the matrix such that

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}$$

Solve the system

$$A\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

2.4 Elementary Matrices

1. **On Elementary Operations-to- Matrices**

(a) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

(b) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \pi & -\pi & \pi & -\pi \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

(c) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 & 3 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

(d) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

(e) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \pi & 2\pi & -2\pi & \pi \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

(f) Let

$$A = \begin{pmatrix} \pi & \pi & \pi & \pi \\ 1 & 2 & -2 & 1 \\ 1-\pi & 1-\pi & 1-\pi & 1-\pi \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 1 \\ 1-\pi & 1-\pi & 1-\pi & 1-\pi \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

(g) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 2 & -2 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

(h) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \sqrt{2} & 2\sqrt{2} & -2\sqrt{2} & \sqrt{2} \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Write down the elementary matrix E such that $EA = B$.

2. **On Inverses of Elementary Matrices**

(a) Compute the inverse of the elementary matrix

$$A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

(b) Compute the inverse of the elementary matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(c) Compute the inverse of the elementary matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \quad \text{where} \quad c \neq 0.$$

(d) Compute the inverse of the elementary matrix

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) Compute the inverse of the elementary matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(f) Compute the inverse of the elementary matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(g) Compute the inverse of the elementary matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where} \quad c \neq 0.$$

3. **Nonsingular Matrices as product of elementary Matrices**

(a) Consider the matrix

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

i. Find a sequence of elementary matrices $E_1, E_2, \ldots$ such that $\cdots E_2 E_1 A = I_3$.

ii. Compute A^{-1}, from above.

(b) Consider the matrix

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad \text{where} \quad abc \neq 0.$$

i. Find a sequence of elementary matrices $E_1, E_2, \ldots$ such that $\cdots E_2 E_1 A = I_3$.

ii. Compute A^{-1}, from above.

(c) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{pmatrix}$$

i. Find a sequence of elementary matrices $E_1, E_2, \ldots$ such that $\cdots E_2 E_1 A = I_3$.

ii. Compute A^{-1}, from above.

(d) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ -1 & -3 & -2 \end{pmatrix}$$

i. Find a sequence of elementary matrices $E_1, E_2, \ldots$ such that $\cdots E_2 E_1 A = I_3$.

ii. Compute A^{-1}, from above.

(e) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & 4 & 7 \end{pmatrix}$$

i. Find a sequence of elementary matrices $E_1, E_2, \ldots$ such that $\cdots E_2 E_1 A = I_3$.

ii. Compute A^{-1}, from above.

Chapter 3

Determinant

3.1 Definitions of Determinant

1. **Determinant of 2×2 matrices**

   (a) Compute the determinant (by any method) of the matrix

   $$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

   (b) Compute the determinant (by any method) of the matrix

   $$A = \begin{pmatrix} \pi & \sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$$

   (c) Compute the determinant (by any method) of the matrix

   $$A = \begin{pmatrix} x & \sqrt{3} \\ \frac{1}{\sqrt{3}} & y \end{pmatrix}$$

   (d) Compute the determinant (by any method) of the matrix

   $$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(e) Compute the determinant (by any method) of the matrix

$$A = \begin{pmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{pmatrix}$$

2. **Determinant of 3×3 matrices**

(a) Use the cofactor method to compute the determinant of the matrix

$$A = \begin{pmatrix} 8 & 7 & 2 \\ 1 & 1 & 3 \\ 9 & 2 & 1 \end{pmatrix}$$

(b) Use the cofactor method to compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & \pi & 1 \\ 1 & 1+\pi & 4 \\ 1 & \pi & 2 \end{pmatrix}$$

(c) Use the cofactor method to compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & x & 1 \\ 1 & 1+x & 4 \\ 1 & x & 2 \end{pmatrix}$$

(d) Use the cofactor method to compute the determinant of the matrix

$$A = \begin{pmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(e) Use the cofactor method to compute the determinant of the matrix

$$A = \begin{pmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

(f) Use the cofactor method to compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}$$

3. **Determinant of** 4×4 **matrices**

   (a) Use the cofactor method to compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   (b) Use the cofactor method to compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

3.2 Computation by Elementary Operation

In this section, reduce the matrix to a triangular matrix, by elementary operations, to compute the determinant.

1. **Triangular Matrices**

   (a) Compute the determinant of the triangular matrix:

$$A = \begin{pmatrix} 2 & \sqrt{3} \\ 0 & 3 \end{pmatrix}$$

   (b) Compute the determinant of the triangular matrix:

$$A = \begin{pmatrix} 2 & a \\ 0 & 3 \end{pmatrix}$$

   (c) Use the theorem on triangular matrices, to determine the determinant of the matrix (*it is a one liner*):

$$A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

(d) Use the theorem on triangular matrices, to determine the determinant of the matrix (*it is a one liner*):

$$A = \begin{pmatrix} x & 0 \\ a & y \end{pmatrix}$$

(e) Use the theorem on triangular matrices, to determine the determinant of the matrix (*it is a one liner*):

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(f) Use the theorem on triangular matrices, to determine the determinant of the matrix (*it is a one liner*):

$$A = \begin{pmatrix} x & 3 & 4 \\ 0 & y & 1 \\ 0 & 0 & x \end{pmatrix}$$

(g) Use the theorem on triangular matrices, to determine the determinant of the matrix (*it is a one liner*):

$$A = \begin{pmatrix} x & a & b \\ 0 & y & c \\ 0 & 0 & x \end{pmatrix}$$

(h) Use the theorem on triangular matrices, to determine the determinant of the matrix (*it is a one liner*):

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) Use the theorem on triangular matrices, to determine the determinant of the matrix (*it is a one liner*):

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ x & b & 0 & 0 \\ y & u & c & 0 \\ z & v & w & d \end{pmatrix}$$

2. **Use Elementary Operations**

(a) Compute the determinant of the matrix A, reducing the matrix to a simpler matrix (usually triangular), by elementary operations:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -2 & -2 & -2 \\ 2 & 2 & 4 & 5 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

(b) Compute the determinant of the matrix A, reducing the matrix to a simpler matrix (usually triangular), by elementary operations:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

(c) Compute the determinant of the matrix A, reducing the matrix to a simpler matrix (usually triangular), by elementary operations:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}$$

(d) Compute the determinant of the matrix A, reducing the matrix to a simpler matrix (usually triangular), by elementary operations:

$$A = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & 1+\sqrt{2} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}$$

(e) Compute the determinant of the matrix A, reducing the matrix to a simpler matrix (usually triangular), by elementary operations:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \pi & \pi & \pi \\ 0 & 0 & 1 & 1 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}$$

(f) Compute the determinant of the matrix A, reducing the matrix to a simpler matrix (usually triangular), by elementary operations:

$$A = \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & y & y & y \\ 0 & 0 & z & z \\ w & w & w & w \end{pmatrix}$$

3.3 Properties of Determinant

1. **On the Product Formula**

   (a) Let A, B be two $n \times n$ matrix. It is given $|A| = 12$ and $|B| = \frac{1}{12}$.

   i. Compute $|BA|$.
   ii. Compute $|B^{-1}A|$.
   iii. Compute $|BA^T|$

   (b) Let A, be a 4×4 matrix and given $|A| = 24$. Let

$$B = \begin{pmatrix} 1 & a & b & c \\ 0 & 2 & x & y \\ 0 & 0 & 3 & z \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

   i. Compute $|BA|$.
   ii. Compute $|B^{-1}A|$.
   iii. Compute $|B^TA|$.

   (c) Let A, be a 4×4 matrix and given $|A| = 2$.

   i. Suppose B is the matrix obtained by multiplying the second row of A by π. Compute the determinant of B.
   ii. Compute $|\pi A|$.

2. **On Nonsingularity** Recall, a square matrix is called nonsingular, if the matrix is invertible.

(a) Suppose

$$A = \begin{pmatrix} 1 & 3 & -3 & -4 \\ 0 & 2 & 2 & 4 \\ 1 & 1 & 3 & 4 \\ 2 & 6 & 2 & 4 \end{pmatrix}$$

Is A nonsingular?

(b) Suppose

$$A = \begin{pmatrix} 1 & 3 & -3 & -4 \\ 0 & 2 & x & y \\ 0 & 0 & 3 & z \\ 1 & 3 & -3 & 0 \end{pmatrix}$$

Is A nonsingular?

(c) Suppose

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & 2 & x & y \\ 0 & 0 & 3 & z \\ 1 & a & b & 4+c \end{pmatrix}$$

Is A nonsingular?

(d) Suppose

$$A = \begin{pmatrix} 0 & 3 & -3 & -4 \\ 0 & 2 & x & y \\ 0 & 0 & 3 & z \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Is A nonsingular?

(e) Suppose

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

For what values of a, b, c, d, the matrix A is nonsingular?

3. **On nonsigularity and uniqueness of solutions** You need not find the explicit solutions of the following systems! Just answer, if the system has unique solutions of not?

4. Consider the linear system

$$\begin{cases} x & +3y & -3z & -4w & = 0 \\ x & +5y & -6z & -6w & = 0 \\ 2x & +6y & -3z & -11w & = 0 \\ 3x & +11y & -9z & -14w & = 0 \end{cases}$$

Does this system have unique solution? (**Remark.** *This system has the trivial solution. Question is, if that is the only one.*)

(a) Consider the linear system

$$\begin{cases} x & +3y & -3z & -4w & = 0 \\ & 2y & +2z & +4w & = -1 \\ x & +y & +3z & +4w & = a \\ 2x & +6y & +2z & +4w & = -1 \end{cases}$$

Does this system have unique solution? (Hint: Use(2a))

(b) Consider the linear system

$$\begin{cases} x & +3y & -3z & -4w & = 1 \\ & 2y & +\lambda z & +\mu w & = 1 \\ & & 3z & +\nu w & = 1 \\ x & +3y & -3z & & = 1 \end{cases}$$

Does this system have unique solution? (Hint: Use (2b))

3.4 Applications of Determinant

1. **On Inverses using cofactor method**

(a) Compute the determinant, the cofactors matrix and the inverse (when exists), of the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

(b) Compute the determinant, the cofactors matrix and the inverse (when exists), of the matrix

$$A = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$$

(c) Compute the determinant, the cofactors matrix and the inverse (when exists), of the matrix

$$A = \begin{pmatrix} a-1 & a \\ a & a+1 \end{pmatrix}$$

(d) Compute the determinant, the cofactors matrix and the inverse (when exists), of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(e) Compute the determinant, the cofactors matrix and the inverse (when exists), of the matrix

$$A = \begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix}$$

2. **Use Cramer's Rule**

(a) Use Cramer's Rule (when possible) to solve the equation

$$\begin{cases} x & -y & +z & = 8 \\ -x & +y & +z & = -8 \\ 2x & +2y & -2z & = 8 \end{cases}$$

(*You can leave your answer in determinant form, without expanding.*)

(b) Use Cramer's Rule (when possible) to solve the equation

$$\begin{cases} x & +3y & -3z & -4w & = 0 \\ x & +5y & -6z & -6w & = 0 \\ 2x & +6y & -3z & -11w & = 0 \\ 3x & +11y & -9z & -14w & = 0 \end{cases}$$

If Cramer's rule does not apply, say so. (*You can leave your answer in determinant form, without expanding.*)

(c) Use Cramer's Rule (when possible) to solve the equation

$$\begin{cases} x & +3y & -3z & -4w & = 0 \\ & 2y & +2z & +4w & = -1 \\ x & +y & +3z & +4w & = a \\ 2x & +6y & +2z & +4w & = -1 \end{cases}$$

If Cramer's rule does not apply, say so. (*You can leave your answer in determinant form, without expanding.*)

(d) Use Cramer's Rule (when possible) to solve the equation

$$\begin{cases} x & +3y & -3z & -4w & = 1 \\ & 2y & +\lambda z & +\mu w & = 1 \\ & & 3z & +\nu w & = 1 \\ x & +3y & -3z & -w & = 1 \end{cases}$$

If Cramer's rule does not apply, say so. (*You can leave your answer in determinant form, without expanding.*)

3. **On area and volume**

(a) Find the area of the triangle passing through the points $(-1, 1), (1, 0), (0, 3)$. Also determine, if the points are collinear.

(b) Find the area of the triangle passing through the points $(-1, -1), (1, 3), (2, 5)$. Also determine if the points are collinear.

(c) Find the area of the triangle passing through the points $(1, 1), (2, 1), (\pi, -1)$. Also determine if the points are collinear.

(d) Find the area of the triangle passing through the points $(1, 1), (2, 4), (3, 9)$. Also determine, if the points are collinear.

(e) Find the volume of the tetrahedron passing through the points$(-1, 1, 0)$, $(1, 0, 0)$, $(0, 3, 0)$, $(1, 1, 1)$. Also determine if the points are coplanar.

(f) Find the volume of the tetrahedron passing through the points$(-1, 1, 1)$, $(2, 4, 8)$, $(-2, 4, -8)$, $(3, 9, 27)$. Also determine if the points are coplanar.

Chapter 4

Vector Spaces

4.1 Vectors in n Spaces $\mathbb{R}^n$

No Homework

4.2 Vector Spaces

1. **The zero and Additive Inverse**

   (a) In $\mathbb{R}^3$, what is the additive inverse of $\mathbf{x} = (\pi, \pi, \pi)$.

   (b) Consider the vector space $V = C(0,1)$ of all the continuous functions $f : (0,1) \longrightarrow \mathbb{R}$.

      i. Describe the zero vector in V.

      ii. Describe the additive inverse of the function $f(x) = e^x$.

      iii. Describe the additive inverse of the constant function $f(x) = 1$.

   (c) Let

$$V = \left\{ \begin{pmatrix} a & b & c & a+b+c \\ x & y & z & x+2y+3z \end{pmatrix} : a,b,c,x,y,z \in \mathbb{R} \right\}$$

i. Convince yourself that V is a subspace of $M_{2\times 4}$, under usual addition and scalar multiplication.
ii. Describe the zero vector in V.
iii. Describe the additive inverse $\mathbf{u} = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{pmatrix}$.

2. Let

$$V = \left\{ \begin{pmatrix} a & b & c & a+b+c+1 \\ x & y & z & x+2y+3z \end{pmatrix} : a, b, c, x, y, z \in \mathbb{R} \right\}$$

Give a reason, why V is not a vector space?

3. Let L be the set of all solutions of the linear system:

$$\begin{cases} 2x + y - z & = 1 \\ x + y - z & = 0 \end{cases}$$

Give a reason, why L is not a vector space?

4.3 Subspaces

The main clue to determine, if something is a subspace or not, is whether expressions used are homogeneous linear or not.

1. **On subspaces of $\mathbb{R}^n$ and $M_{m\times n}$.**

(a) Verify, if the set

$$W = \{(x, y, x + 2y) : x, y \in \mathbb{R}\} \quad \text{is a subspace of } \mathbb{R}^3 \text{ or not?}$$

Solution: , Here all three coordinates are homogeneous in x, y. So, I expect W to be a subspace, which I prove, by checking three conditions.

i. With $x = y = 0$, the zero vector $\mathbf{0} = (0, 0, 0) \in W$.
So, W is nonempty.

ii. Let $\mathbf{u} = (x_1, y_1, z_1), \mathbf{v} = (x_2, y_2, z_2) \in W$. So,

$$\begin{cases} z_1 = x_1 + 2y_1, \\ z_2 = x_2 + 2y_2 \end{cases} \quad \text{and} \quad \mathbf{u}+\mathbf{v} = (x_1+x_2, y_1+y_2, z_1+z_2).$$

Now,

$$z_1 + z_2 = (x_1 + 2y_1) + (x_2 + 2y_2) = (x_1 + x_2) + 2(y_1 + y_2)$$

So, $\mathbf{u} + \mathbf{v} \in W$. So, W is closed under addition.

iii. Now let $\mathbf{u} = (x_1, y_1, z_1) \in W$ and $c \in \mathbb{R}$. As above, $z_1 = x_1 + 2y_1$. Also,

$$c\mathbf{u} = (cx_1, cy_1, cz_1). \quad \text{We have} \quad cz_1 = (cx_1) + 2(cy_1)$$

So, $c\mathbf{u} \in W$. So, W is closed under scalar multiplication..

So, W is a subspace.

(b) Verify, if the set

$$W = \{(x + y, x - y, 0) : x, y \in \mathbb{R}\} \quad \text{is a subspace of } \mathbb{R}^3 \text{ or not?}$$

Solution: , Here all three coordinates are homogeneous in x, y. So, I expect W to be a subspace, which I prove, by checking three conditions.

i. With $x = y = 0$, the zero vector $\mathbf{0} = (0, 0, 0) \in W$. So, W is nonempty.

ii. Let $\mathbf{u} = (x_1, y_1, z_1), \mathbf{v} = (x_2, y_2, z_2) \in W$. So,

$$\text{for some } t_1, s_1 \in \mathbb{R} \quad \begin{cases} x_1 = t_1 + s_1 \\ y_1 = t_1 - s_1 \\ z_1 = 0 \end{cases}$$

and

$$\text{for some } t_2, s_2 \in \mathbb{R} \quad \begin{cases} x_2 = t_2 + s_2 \\ y_2 = t_2 - s_2 \\ z_2 = 0 \end{cases}$$

Also,

$$\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

And,

$$\begin{cases} x_1 + x_2 = (t_1 + t_2) + (s_1 + s_2) \\ y_1 + y_2 = (t_1 + t_2) - (s_1 + s_2) \\ z_1 + z_2 = 0 \end{cases}$$

So, $\mathbf{u} + \mathbf{v} \in W$. So, W is closed under addition.

iii. Let $\mathbf{u} = (x, y, z) \in W$ and $c \in \mathbb{R}$. So,

$$\text{for some } t, s \in \mathbb{R} \quad \begin{cases} x = t + s \\ y = t - s \\ z = 0 \end{cases}$$

Now,

$$c\mathbf{u} = (cx, cy, z) \quad \text{and} \quad \begin{cases} cx = ct + cs \\ y = ct - cs \\ z = 0 \end{cases}$$

So, $c\mathbf{u} \in W$. So, W is closed under scalar multiplication.

So, W is a subspace.

(c) Verify, if the set

$$W = \{(x, y, x + \pi y) : x, y \in \mathbb{R}\} \quad \text{is a subspace of } \mathbb{R}^3 \text{ or not?}$$

Solution: Similar to (1a).

(d) Verify, if the set

$$W = \{(x, y, x + \pi y + 13) : x, y \in \mathbb{R}\} \quad \text{is a subspace of } \mathbb{R}^3 \text{ or not?}$$

Solution: Since the last coordinate is not homogeneous, I do not expect it to be a subspace. To prove this, not

$$(x, y, x + \pi y + 13) \neq (0, 0, 0) \quad \forall x, y \in \mathbb{R}$$

So, the zero vector $\mathbf{0} = (0, 0, 0) \notin W$. So, W is not a subspace.

(e) Verify, if the set

$$W = \{(y^2 + z^2, y, z) : y, z \in \mathbb{R}\} \quad \text{is a subspace of } \mathbb{R}^3 \text{ or not?}$$

Solution: Note first coordinate is not linear. So, I do not expect it to be a subspace.
Actually, $\mathbf{0} = (0, 0, 0) \in W$. So, we have to try something else.
Now, $(1, 1, 0) \in W$. $2(1, , 1, 0) = (2, 2, 0) \notin W$.
So, W is not closed under scalar multiplication. So, W is not a subspace.

(f) Verify, if the set

$$W = \{(0, y, z) : y, z \in \mathbb{R}\} \quad \text{is a subspace of } \mathbb{R}^3 \text{ or not?}$$

(g) Verify, if the set

$$W = \left\{ \begin{pmatrix} x & y \\ 0 & x + \pi y \end{pmatrix} : x, y \in \mathbb{R} \right\} \quad \text{is a subspace of } M_{2\times 2} \text{ or not?}$$

Solution: Similar to (1a).

(h) Verify, if the set

$$W = \left\{ \begin{pmatrix} x & y \\ 0 & x + \pi y + 13 \end{pmatrix} : x, y \in \mathbb{R} \right\} \quad \text{is a subspace of } M_{2\times 2} \text{ or not?}$$

Solution: Similar to (1d).

(i) Verify, if the set

$$W = \{(x, y, z) : x, y, z \in \mathbb{R}, z \text{ is an integer}\} \quad \text{is a subspace of } \mathbb{R}^3 \text{ or not?}$$

Remark. Intuitively, for W to be a subspace, each coordinate should be a homogenous linear polynomial, in some free variables, like x, y etc.

2. **On subspaces of $C(-1, 1)$**

Let $V = C(-1, 1)$ be the vector space of all continuous real valued functions on on the interval $(-1, 1)$, with usual addition and scalar multiplication..

Clue: *If believe, this is the only example we are doing that does not have coordinates. For this problem main clue is whether it is defined by vanishing of functions, on a point or subset.* Also, note that the constant zero function

$$c_0 : (-1, 1) \to \mathbb{R} \quad \text{defined by} \quad c_0(x) = 0 \quad \forall\, x \in (-1, 1)$$

is **the zero** of the vector space $V = C(-1, 1)$.

(a) Verify, if the set

$$W = \{f \in V : f(0) = 0\} \quad \text{is a subspace of } V \text{ or not?}$$

Solution: W is defined by vanishing at the point $x = 0$. So, I expect that W is a subspace of V.

i. Note $c_0(0) = 0$. So, $c_0 \in W$. So, W is nonempty.
ii. Let $f, g \in W$. Then, $f(0) = 0$ and $g(0) = 0$.
So, $(f + g)(0) = f(0) + g(0) = 0$. So, $f + g \in W$.
So, W is closed under addition.
iii. Let $f \in W$ and $c \in \mathbb{R}$. Then, $f(0) = 0$.
So, $(cf)(0) = cf(0) = 0$. So, $cf \in W$.
So, W is closed under scalar multiplication.

So, W is a subspace.

(b) Verify, if the set

$$W = \{f \in V : f(0) = 1\} \quad \text{is a subspace of } V \text{ or not?}$$

Solution: W is not defines by vanishing. So, I do not expect it to be a subspace.
To prove this, note that the zero of V, $c_0 \notin W$. So, W is not a subspace.

(c) Verify, if the set

$$W = \left\{f \in V : f(x) = 0 \ \ \forall\ -\frac{1}{2} \le x \le \frac{1}{2}\right\} \quad \text{is a subspace of } V \text{ or not?}$$

Solution: Here W is defined by vanising on the subset $\left[-\frac{1}{2}, \frac{1}{2}\right]$.
So, I expect it to be a subspace.
The proof is exactly similar to (2a).

(d) Verify, if the set

$$W = \left\{ f \in V : f(x) = -1 \ \ \forall \ -\frac{1}{2} \leq x \leq \frac{1}{2} \right\} \quad \text{is a subspace of } V \text{ or not?}$$

Solution: Note the zero of V, $c_0 \notin W$. So, W is not a subspace.

3. **On subspaces of P**

   Let **P** be the vector space of all polynomials, with real coefficients, with usual addition and scalar multiplication.

   (a) Verify, if the set

   $$W = \{ f \in \mathbf{P} : f(0) = 0 \} \quad \text{is a subspace of } \mathbf{P} \text{ or not?}$$

   Solution: The proof is exactly similar to (2a).

   (b) Verify, if the set

   $$W = \{ f \in \mathbf{P} : f(0) = 1 \} \quad \text{is a subspace of } \mathbf{P} \text{ or not?}$$

4.4 Spanning and Linear Independence

1. **On Linear combination**

   (a) Let $S = \{(-1, -1)\}$. Can we write $(1, 2)$ as a linear combination of the vectors in S?

   (b) Let $S = \{(-1, -1, -1)\}$. Can we write $(1, 2, 0)$ as a linear combination of the vectors in S?

   (c) Let $S = \{(1, 1, 1), (-1, 1, 1)\}$. Can we write $(2, 2, 2)$ as a linear combination of the vectors in S?

   (d) Let $S = \{(1, 1, 1), (-1, 1, 1)\}$. Can we write $(2, 0, 0)$ as a linear combination of the vectors in S?

   (e) Let $S = \{(1, 1, 1), (-1, 1, 1)\}$. Can we write $(2, 0, 0)$ as a linear combination of the vectors in S?

   (f) Let $S = \{(1, 1, 1), (-1, 1, 1)\}$. Can we write $(2, 4, 4)$ as a linear combination of the vectors in S?

2. **On the spanning set**

   (a) Let $S = \{(1,1,1), (-1,1,1)\}$. Describe the spanning set of S.
   Solution; I will solve this one, for guidance for the next few.:

   $$span(S) = \{a(1,1,1)+b(-1,1,1) : a,b \in \mathbb{R}\} = \{(a-b, a+b, a+b) : a,b \in \mathbb{R}\}$$

   (b) Let $S = \{(1,1,1)\}$. Describe the spanning set of S. (*Try to visualize it, geometrically*!)

   (c) Let $S = \{(1,1,0), (0,0,1)\}$. Describe the spanning set of S. (*Try to visualize it, geometrically*!)

   (d) Let $S = \{(1,0,0), (0,1,0)\}$. Describe the spanning set of S. (*Try to visualize it, geometrically*!)

   (e) Let $S = \{(1,1,0), (1,-1,0)\}$. Describe the spanning set of S. (*Try to visualize it, geometrically*!)

3. **On the spanning** $\mathbb{R}^n$

   (a) Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$. Does S span $\mathbb{R}^3$.

   (b) Let $S = \{(1,0,0,0), (0,1,0,0), (0,0,1), (0,0,0,1)\}$. Does S span $\mathbb{R}^4$.

   (c) Let $S = \{(1,1,1), (1,-1,1), (1,1,-1)\}$. Does S span $\mathbb{R}^3$.

   (d) Let $S = \{(1,0,1), (1,2,1), (1,2,2), (13,17,19)\}$. Does S span $\mathbb{R}^3$.

   (e) Let $S = \{(1,1,1), (1,2,1), (0,1,0), (3,4,3)\}$. Does S span $\mathbb{R}^3$.

4. **On the spanning** $M_{m\times n}$

   (a) Let

   $$S = \left\{ \mathbf{e}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{e}_4 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

   Does S span $M_{2\times 2}$?

   (b) Let

   $$S = \left\{ \mathbf{e}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{e}_4 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

   Does S span $M_{2\times 2}$?

5. **On the spanning $\mathbf{P}_n$**

   Let $\mathbf{P}_2$ denote the vector space of all polynomials, of degree ≤ 2.

   (a) Let $S = \{x^2, x, 1\}$. Does S span $\mathbf{P}_2$.
   Solution: A polynomial $f(x) \in \mathbf{P}_2$ has the form
   $f(x) = ax^2 + bx + c$.
   Any such polynomial is a linear combination of elements in $S = \{x^2, x, 1\}$. So, S spans $\mathbf{P}_2$.

   (b) Let $S = \{(x^2, x, 1, x^2 + x + 1\}$. Does S span $\mathbf{P}_2$.
   Solution: A polynomial $f(x) \in \mathbf{P}_2$ has the form
   $f(x) = ax^2 + bx + c$.
   Question is whether we can write such a polynomial

   $$f(x) = ax^2 + bx + c = \alpha(x^2) + \beta(x) + \gamma(1) + \delta(x^2 + x + 1)$$

   In this case, it is easier than other problems, because we can take

   $$\alpha = a, \quad \beta = b, \quad \gamma = x, \quad \delta = 0$$

   So, S spans $\mathbf{P}_2$.

   (c) Let $S = \{x^2 + x + 1, x^2 - x + 1, x^2 + x - 1\}$. Does S span $\mathbf{P}_2$.
   Solution: A polynomial $f(x) \in \mathbf{P}_2$ has the form
   $f(x) = ax^2 + bx + c$.
   Question is whether we can write such a polynomial

   $$f(x) = ax^2 + bx + c = \alpha(x^2 + x + 1) + \beta(x^2 - x + 1) + \gamma(x^2 + x - 1) \Longrightarrow$$

   $$ax^2 + bx + c = (\alpha + \beta + \gamma)x^2 + (\alpha - \beta + \gamma)x + (\alpha + \beta - \gamma) \Longrightarrow$$

   $$\begin{pmatrix} \alpha + \beta + \gamma \\ \alpha - \beta + \gamma \\ \alpha + \beta - \gamma \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

   The system has a solution for any a, b, c. So, S spans $\mathbf{P}_2$.

   (d) Let $S = \{x^2 + 1, x^2 + 2x + 1, x^2 + 2x + 2, 13x^2 + 17x + 19)\}$. Does S span $\mathbf{P}_2$.

Solution: A polynomial $f(x) \in \mathbf{P}_2$ has the form
$f(x) = ax^2 + bx + c$.
Question is whether we can write such a polynomial
$f(x) = ax^2 + bx + c$

$= \alpha(x^2+1)+\beta(x^2+2x+1)+\gamma(x^2+2x+2)+\delta(13x^2+17x+19) \Longrightarrow$

$$\begin{pmatrix} \alpha+\beta+\gamma+13\delta \\ 2\beta+2\gamma+17\delta \\ \alpha+\beta+2\gamma+19\delta \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 13 \\ 0 & 2 & 2 & 17 \\ 1 & 1 & 2 & 19 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Do row Echelon to the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 13 & a \\ 0 & 2 & 2 & 17 & b \\ 1 & 1 & 2 & 19 & c \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 1 & 13 & a \\ 0 & 2 & 2 & 17 & b \\ 0 & 0 & 1 & 6 & c-a \end{pmatrix}$$

Now, we see that the system has a solution for any a, b, c. So, S spans $\mathbf{P}_2$.

(e) Let $S = \{x^2 + x + 1, x^2 + 2x + 1, x, 3x^2 + 4x + 3\}$. Does S span $\mathbf{P}_2$.
Solution: A polynomial $f(x) \in \mathbf{P}_2$ has the form
$f(x) = ax^2 + bx + c$.
Question is whether we can write such a polynomial
$f(x) = ax^2 + bx + c$

$= \alpha(x^2 + x + 1) + \beta(x^2 + 2x + 1) + \gamma(x) + \delta(3x^2 + 4x + 3) \Longrightarrow$

$$\begin{pmatrix} \alpha+\beta+3\delta \\ \alpha+2\beta+\gamma+4\delta \\ \alpha+\beta++3\delta \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Consider the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 3 & a \\ 1 & 2 & 1 & 4 & b \\ 1 & 1 & 0 & 3 & c \end{pmatrix}$$

Reduce it to row Echelon:

$$\begin{pmatrix} 1 & 1 & 0 & 3 & a \\ 0 & 1 & 1 & 1 & b-a \\ 0 & 0 & 0 & 0 & c-a \end{pmatrix}$$

The system does not have solution, if $c - a \neq 0$. So, S does not span $\mathbf{P}_2$.

6. **On Linear Independence vectors in $\mathbb{R}^n$**

   (a) Let $S = \{(1, 1), (\pi, \pi)\}$. Is S linearly independent of not?

   (b) Let $S = \{(1, 1), (\pi, 0)\}$. Is S linearly independent of not?

   (c) Let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Is S linearly independent of not?

   (d) Let $S = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1), (0, 0, 0, 1)\}$. Is S linearly independent of not?

   (e) Let $S = \{(1, 1, 1), (1, -1, 1), (1, 1, -1)\}$. Is S linearly independent of not?

   (f) Let $S = \{(1, 0, 1), (1, 2, 1), (1, 2, 2), (13, 17, 19)\}$. Is S linearly independent of not?

   (g) Let $S = \{(1, 1, 1), (1, 2, 1), (0, 1, 0), (3, 4, 3)\}$. Is S linearly independent of not?

7. **On Linear Independence of vectors in $M_{m \times n}$**

   (a) Let

   $$S = \left\{ \mathbf{e}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{e}_4 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

   Is S linearly independent of not?

(b) Let

$$S = \left\{ \mathbf{e}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{e}_4 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

Is S linearly independent of not?

8. **On Linear Independence of vectors in $\mathbf{P}_n$**

Let $\mathbf{P}_2$ denote the vector space of all polynomials, of degree ≤ 2.

(a) Let $S = \{x^2, x, 1\}$. Is S linearly independent of not?
Solution: For $a, b, c \in \mathbb{R}$, we have

$$ax^2 + bx + c(1) = 0 \Longrightarrow ax^2 + bx + c = 0 \Longrightarrow a = b = c = 0$$

So, they are linearly independent.

(b) Let $S = \{(x^2, x, 1, x^2 + x + 1\}$.Is S linearly independent of not?
Solution: In this case, we can write down the last vector x^2+x+1, as linear combination of the otheres:

$$x^2 + x + 1 = 1(x^2) + 1(x) + 1(1)$$

So, they are linearly dependent

(c) Let $S = \{x^2+x+1, x^2-x+1, x^2+x-1\}$. Is S linearly independent of not?
Solution: For $a, b, c \in \mathbb{R}$, we have

$$a(x^2 + x + 1) + b(x^2 - x + 1) + c(x^2 + x - 1) = 0 \Longrightarrow$$

$$(a + b + c)x^2 + (a - b + c)x + (a + b - c) = 0 \Longrightarrow$$

$$\begin{pmatrix} a + b + c \\ a - b + c \\ a + b - c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow a = b = c = 0$$

So, they are linearly independent.

(d) Let $S = \{x^2 + 1, x^2 + 2x + 1, x^2 + 2x + 2, 13x^2 + 17x + 19\}$. Is S linearly independent of not?
Solution: For $a, b, c, d \in \mathbb{R}$, we have

$$a(x^2+1)+b(x^2+2x+1)+c(x^2+2x+2)+d(13x^2+17x+19) = 0 \quad \Longrightarrow$$

$$(a+b+c+13d)x^2+(2b+2c+17d)x+(a+b+2c+19d)=0 \implies$$

$$\begin{pmatrix} a+b+c+13d \\ 2b+2c+17d \\ a+b+2c+19d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This system has **nonzero** solutions for a, b, c, d. So, the are linearly dependent.

(e) Let $S = \{x^2 + x + 1, x^2 + 2x + 1, x, 3x^2 + 4x + 3\}$. Is S linearly independent of not?

Solution: For $a, b, c, d \in \mathbb{R}$, we have

$$a(x^2+x+1)+b(x^2+2x+1)+c(x)+d(3x^2+4x+3)=0 \implies$$

$$(a+b+3d)x^2+(a+2b+c+4d)x+(a+b+3d)=0 \implies$$

$$\begin{pmatrix} a+b+3d \\ a+2b+c+4d \\ a+b+3d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This system has **nonzero** solutions for a, b, c, d. So, the are linearly dependent.

9. **On Linearly Dependent of vectors**

(a) Let $S = \{(1, 2), (2, 1), (2, 2)\}$. We know, S is a linearly dependent set. By Theorem 4.2.2, one of the vectors, is linear combination of the rest. Write down, one as linear combination of the rest.

Solution: Actually,

$$(2, 2) = (2/3)(1, 2) + (2/3)(2, 1)$$

They are dependent, because one of them is linear combination of the others.

(b) Let $S = \{(1, 1, 1), (1, -1, 1), (1, 1, -1), (6, 2, 0)\}$. We know, S is a linearly dependent set. By Theorem 4.2.2, one of the vectors, is linear combination of the rest. Write down, one as linear combination of the rest.

Solution: Actually,

$$(6, 2, 0) = (1, 1, 1) + 2(1, -1, 1) + 3(1, 1, -1)$$

They are dependent, because one of them is linear combination of the others.

(c) Let $S = \{(1,1,0),(1,0,1),(0,1,-1),(6,6,0)\}$. We know, S is a linearly dependent set. By Theorem 4.2.2, one of the vectors, is linear combination of the rest. Write down, one as linear combination of the rest.

Solution: Actually,

$$(6,6,0) = 3(1,1,0) + 3(1,0,1) + 3(0,1,-1)$$

They are dependent, because one of them is linear combination of the others.

4.5 Basis and Dimension

1. **On failure to be a basis**

   Answer for these should exactly one sentence. Assume $a, b, c \in \mathbb{R}$.

   (a) Consider the subset $S = \{(1,1,0),(17,113,120),(0,1,\sqrt{7}),(a,2,c))\} \subseteq \mathbb{R}^3$. Give a reason, why S is not a basis of $\mathbb{R}^3$.
   Solution: $\dim \mathbb{R}^3 = 3$.

   (b) Consider the subset $S = \{(\pi, e, \sqrt{7}),(a,2,c))\} \subseteq \mathbb{R}^3$. Give a reason, why S is not a basis of $\mathbb{R}^3$.
   Solution: $\dim \mathbb{R}^3 = 3$.

   (c) Consider the subset $S = \{(1,0,0,0),(0,1,0,0,),(0,1,0,0)\} \subseteq \mathbb{R}^4$. Give a reason, why S is not a basis of $\mathbb{R}^4$.
   Solution: $\dim \mathbb{R}^4 = 4$.

   (d) Consider the subset
   $S = \{(1,\pi,\pi^2,\pi^3),(1,e,e^2,e^3,),(1,2,4,8),(1,3,9,27),(1,-1,1,-1)\} \subseteq \mathbb{R}^4$. Give a reason, why S is not linearly independent?

   (e) Consider the subset
   $S = \{(1,\pi,\pi^2,\pi^3),(1,e,e^2,e^3,),(1,2,4,8)\} \subseteq \mathbb{R}^4$. Give a reason, why S does not span $\mathbb{R}^4$?

   (f) Let $\mathbf{P}_3$ be the vector space of all the polynomials, with real coefficients, of degree ≤ 3. Consider the subset
   $S = \{x + x^3, 17 + 13x + 10x^3, 1 + \sqrt{7}x + ax^3, x^2 + x^3, x^3\} \subseteq \mathbf{P}_3$. Give a reason, why S is not a basis of $\mathbf{P}_3$.

Solution: $\dim \mathbf{P}_3 = 4$.

(g) Let $\mathbf{P}_3$ be the vector space of all the polynomials, with real coefficients, of degree ≤ 3. Consider the subset $S = \{x + x^3, x^2 + x^3, x^3\} \subseteq \mathbf{P}_3$. Give a reason, why S is not a basis of $\mathbf{P}_3$.

Solution: $\dim \mathbf{P}_3 = 4$.

(h) Let $M_{2\times3}$ be the vector space of all matrices of size 2×3, with real coefficients. Consider the subset

$$S = \left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \right\} \subseteq M_{2\times3}$$

Give a reason, why S is not a basis of $M_{2\times3}$.

Solution: $\dim M_{2\times3} = 6$.

2. **Determine, if the set is a basis**

In the following problems, the dimension of the vector space and the cardinality of S would match. One way to get the answer is to check the determinant of the matrix formed by them.

(a) Let $S = \{(1, -1, 1), (1, \sqrt{2}, 2), (1, \sqrt{3}, 3)\} \subset \mathbb{R}^3$. Is S a basis of $\mathbb{R}^3$?

Solution: We have

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & \sqrt{2} & 2 \\ 1 & \sqrt{3} & 3 \end{vmatrix} = 2.096 \neq 0$$

So, S is a basis of $\mathbb{R}^3$.

(b) Let $S = \{(1, -1, 1, -1), (1, 1, 1, 1), (1, \sqrt{2}, 2, 2\sqrt{2}), (1, \sqrt{3}, 3, 3\sqrt{3})\} \subset \mathbb{R}^4$. Is S a basis of $\mathbb{R}^4$?

Solution: We have

$$\begin{vmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & \sqrt{2} & 2 & 2\sqrt{2} \\ 1 & \sqrt{3} & 3 & 3\sqrt{3} \end{vmatrix} = 1.27 \neq 0$$

3. **Span and Dimension**

(a) Let $S = \{(1,1,1)\}$ and $V = span(S) \subseteq \mathbb{R}^3$. Find $\dim(V)$, and give basis of V.

Solution: A basis of V is given by $\{(1,1,1)\}$. So, $\dim V = 1$.

(b) Let $S = \{(1,1,1),(1,-1,1),(1,0,1)\}$ and $V = span(S) \subseteq \mathbb{R}^3$. Find $\dim(V)$, and give basis of V.

Solution: Form the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{Do Row Echelon (ref)} \quad \Longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, a basis of $Span(S) = \{(1,1,1),(0,1,0)\}$ and $\dim Span(S) = 2$.

(c) Let $S = \{(1,1,1),(1,-1,1),(\pi,0,\pi)\}$ and $V = span(S) \subseteq \mathbb{R}^3$. Find $\dim(V)$, and give basis of V.

Solution: Form the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ \pi & 0 & \pi \end{pmatrix} \quad \text{Do Row Echelon (ref)} \quad \Longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, a basis of $Span(S) = \{(1,1,1),(0,1,0)\}$ and $\dim Span(S) = 2$.

(d) Let $S = \{(1,1,1,1),(1,-1,1,-1),(1,0,1,0),(0,1,0,1)\}$ and $V = span(S) \subseteq \mathbb{R}^4$. Find $\dim(V)$, and give basis of V.

(e) Let $S = \{(1,-1,1,-1),(1,1,1,1),(1,2,4,8),(1,3,9,27)\}$ and $V = span(S) \subseteq \mathbb{R}^4$. Find $\dim(V)$, and give basis of V.

(f) With $a,b,c \in \mathbb{R}$ and let $S = \{(1,a,a,a),(0,1,b,b),(0,0,1,c)\}$ and $V = span(S) \subseteq \mathbb{R}^4$. Find $\dim(V)$, and give basis of V.

Solution: Here a,b,c are given real numbers. Form the matrix

$$\begin{pmatrix} 1 & a & a & a \\ 0 & 1 & b & b \\ 0 & 0 & 1 & c \end{pmatrix}$$

The matrix is, already, in row Echelon form. So, a basis of $span(S) = \{(1,a,a,a),(0,1,b,b),(0,0,1,c)\}$ and $\dim Span(S) = 3$.

4.6 Rank and Nullity

There is, essentially, one type of problems in the section.

1. Let

$$A = \begin{pmatrix} -7 & 3 & 2 \\ 12 & 2 & 3 \\ 5 & 5 & 5 \end{pmatrix}$$

   (a) Give a basis of the row space of A
   (b) Find $rank(A)$
   (c) Find $nullity(A)$.
   (d) Give a basis of the null space $N(A)$.
   (e) Give basis of the column space of A
   (f) Give a basis of the null space $N(A^T)$.

   Solution: Reduce the matrix to essentially row Echelon form:

$$\begin{pmatrix} -7 & 3 & 2 \\ 5 & 5 & 5 \\ 12 & 2 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} -7 & 3 & 2 \\ 5 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -7 & 3 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -7 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 10 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$

   (a) Give a basis of the row space of A
   A basis is

$$\{(1,1,1),(0,10,9)\}$$

   (b) Find $rank(A) = 2$
   (c) Find $nullity(A) = 3 = 2 = 1$.
   (d) Give a basis of the null space $N(A)$.
   The Null space

$$N(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} 1 & 1 & 1 \\ 0 & 10 & 9 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$N(A) = \left\{ \begin{pmatrix} -\frac{1}{10}t \\ -\frac{9}{10}t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

With $t = 1$, a basis for $N(A)$ is

$$\left\{ \begin{pmatrix} -\frac{1}{10} \\ -\frac{9}{10} \\ 1 \end{pmatrix} \right\}$$

(e) Give basis of the column space of A
Do the same calculation with

$$B = A^T = \begin{pmatrix} -7 & 12 & 5 \\ 3 & 2 & 5 \\ 2 & 3 & 5 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 16 & 15 \\ 3 & 2 & 5 \\ 2 & 3 & 5 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -1 & 16 & 15 \\ 0 & 50 & 50 \\ 0 & 35 & 35 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 16 & 15 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 16 & 15 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

. So, the basis of the column space:

$$\left\{ (-1, 16, 15)^T, (0, 1, 1)^T \right\}$$

(f) Give a basis of the null space $N(A^T)$.
The $N(A^T)$ is given by

$$\begin{pmatrix} -1 & 16 & 15 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$N(A^T) = \left\{ \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} \right\}$$

With $t = 1$, a basis for $N(A^T)$ is

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

2. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 5 & 4 & 2 \\ -7 & 3 & 2 & 0 \end{pmatrix}$$

   (a) Give a basis of the row space of A

   (b) Find $rank(A)$

   (c) Find $nullity(A)$.

   (d) Give a basis of the null space $N(A)$.

   (e) Give basis of the column space of A

   (f) Give a basis of the null space $N(A^T)$.

Solution: (*I do not mind using TI Calculator, unless it give nonterminating decimals that I cannot read.*) We do essential row Echelon to A:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -3 \\ 0 & 10 & 9 & 7 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 10 & 9 & 7 \\ 0 & 0 & -1 & -3 \end{pmatrix}$$

   (a) A Basis of the row space is:

$$\{(1,1,1,1), (0,10,9,7), (0,0,-1,-3)\}$$

   (b) $rank(A) = 3$

   (c) $nullity(A) = 4 - 3 = 1$.

   (d) The null space $N(A)$ is given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 10 & 9 & 7 \\ 0 & 0 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{So,} \qquad N(A) = \left\{ \begin{pmatrix} \frac{27}{10}t \\ \frac{2}{10}t \\ -3t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

With $t = 1$, a basis of $N(A)$ is

$$\left\{ \begin{pmatrix} \frac{27}{10} \\ \frac{2}{10} \\ -3 \\ 1 \end{pmatrix} \right\}$$

(e) Regarding column space of A, we do that same with

$$B = A^T = \begin{pmatrix} 1 & 5 & -7 \\ 1 & 5 & 3 \\ 1 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

We reduce it to essential row Echelon:

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 5 & -7 \\ 1 & 5 & 3 \\ 1 & 4 & 2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -7 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -7 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -7 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & -7 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So, a basis of the row space of A^T is these three nonzero rows. So, a basis of the column space is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(f) For basis of the null space $N(A^T)$, we solve

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$N(A^T) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

So, the zero vector space $N(A^T)$ has empty basis.

3. Let

$$A = \begin{pmatrix} 3 & 15 & -1 \\ 1 & 4 & 2 \\ 1 & 2 & 0 \\ 1 & 5 & 3 \end{pmatrix}$$

   (a) Give a basis of the row space of A
   (b) Find $rank(A)$
   (c) Find $nullity(A)$.
   (d) Give a basis of the null space $N(A)$.
   (e) Give basis of the column space of A
   (f) Give a basis of the null space $N(A^T)$.

4. Let

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 4 & 3 & 3 & 3 & 3 \\ 4 & 3 & 1 & 1 & 1 \end{pmatrix}$$

   (a) Give a basis of the row space of A
   (b) Find $rank(A)$
   (c) Find $nullity(A)$.
   (d) Give a basis of the null space $N(A)$.
   (e) Give basis of the column space of A
   (f) Give a basis of the null space $N(A^T)$.

Chapter 5

Eigenvalues and Eigenvectors

5.1 Eigen Values and Eigen Vectors

1. Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

   (a) Write down the characteristic equation of A

   (b) Find all the eigenvalues of A.

   (c) For each eigenvalue λ, compute the eigenspace $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

   Solution: The characteristic polynomial is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda$$

   So, the characteristic equation is $\lambda^2 - 2\lambda = 0$ and the eigen values are $\lambda = 0, 2$.

   (a) Eigen vectors of $\lambda = 2$ is given by

$$(\lambda I - A)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = t \\ y = -x = -t \end{cases}$ So, the Eigen space

$$E(2) = \left\{ \begin{pmatrix} -t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Taking $t = 1$, a basis of $E(2)$ is $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$. So, $\dim E(2) = 1$.

(b) Eigen vectors of $\lambda = 0$ is given by

$$(\lambda I - A) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = t \\ y = x = t \end{cases}$ So, the Eigen space

$$E(0) = \left\{ \begin{pmatrix} t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Taking $t = 1$, a basis of $E(0)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. So, $\dim E(0) = 1$.

2. Let

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigenvalues of A.

(c) For each eigenvalue λ, compute the eigenspace $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

Solution: The characteristic polynomial is $\det(\lambda I - A) =$

$$\begin{vmatrix} \lambda - 3 & 0 & 1 \\ 0 & \lambda - 2 & 0 \\ 1 & 0 & \lambda - 3 \end{vmatrix} = - \begin{vmatrix} 0 & \lambda - 2 & 0 \\ \lambda - 3 & 0 & 1 \\ 1 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 2)^2(\lambda - 4)$$

So, the characteristic equation is $(\lambda - 2)^2(\lambda - 4) = 0$ and the eigen values are $\lambda = 2, 4$.

(a) Eigen vectors of $\lambda = 2$ is given by

$$(\lambda I - A)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = t \\ z = x = t \\ y = s \end{cases}$ So, the Eigen space

$$E(2) = \left\{ \begin{pmatrix} t \\ s \\ t \end{pmatrix} : t, s \in \mathbb{R} \right\}$$

Taking $t = 1, s = 0$ and $t = 0, s = 1$, respectively, a basis of $E(2)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad \text{and} \qquad \dim E(2) = 2$$

Remark. *Note the eigenvalue $\lambda = 2$ has multiplicity 2. So,* $\dim E(2) \leq 2$ (which we did not prove). *In this case, we did get two independent basis. It would be possible to have* $\dim E(2) = 1$, *in other problems.*

(b) Eigen vectors of $\lambda = 4$ is given by

$$(\lambda I - A)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = t \\ z = -x = -t \\ y = 0 \end{cases}$ So, the Eigen space

$$E(4) = \left\{ \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Taking $t = 1$, a basis of $E(4)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} \qquad \text{and} \qquad \dim E(4) = 1$$

3. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

   (a) Write down the characteristic equation of A
   (b) Find all the eigenvalues of A.
   (c) For each eigenvalue λ, compute the eigenspace $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

Solution: The characteristic polynomial is $\det(\lambda I - A) =$

$$\begin{vmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda-2 & 0 \\ 1 & -2 & \lambda-1 \end{vmatrix} = -\begin{vmatrix} 0 & \lambda-2 & 0 \\ \lambda-1 & -2 & 1 \\ 1 & -2 & \lambda-1 \end{vmatrix} = \lambda(\lambda-2)^2$$

So, the characteristic equation is $\lambda(\lambda-2)^2 = 0$ and the eigen values are $\lambda = 0, 2$.

(a) Eigen vectors of $\lambda = 2$ is given by

$$(\lambda I - A)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = t \\ y = s \\ z = 2y - x = 2s - t \end{cases}$ So, the Eigen space

$$E(2) = \left\{ \begin{pmatrix} t \\ s \\ 2s-t \end{pmatrix} : t, s \in \mathbb{R} \right\}$$

Taking $t = 1, s = 0$ and $t = 0, s = 1$, respectively, a basis of $E(2)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\} \qquad \text{and} \qquad \dim E(2) = 2$$

Remark. *Note the eigenvalue $\lambda = 2$ has multiplicity 2. So,* $\dim E(2) \leq 2$ (which we did not prove). *In this case, we did get two independent basis. It would be possible to have* $\dim E(2) = 1$, *in other problems.*

(b) Eigen vectors of $\lambda = 0$ is given by

$$(\lambda I - A)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ 0 & -2 & 0 \\ 1 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = t \\ y = 0 \\ z = x + 2y = t \end{cases}$ So, the Eigen space

$$E(0) = \left\{ \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Taking $t = 1$, a basis of $E(0)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \text{and} \qquad \dim E(0) = 1$$

4. Let

$$A = \begin{pmatrix} 1 & 2 & -6 \\ -2 & 5 & -6 \\ -2 & 2 & -3 \end{pmatrix}$$

(a) Write down the characteristic equation of A

(b) Find all the eigenvalues of A.

(c) For each eigenvalue λ, compute the eigenspace $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

Solution: The characteristic polynomial is $\det(\lambda I - A) =$

$$\begin{vmatrix} \lambda-1 & -2 & 6 \\ 2 & \lambda-5 & 6 \\ 2 & -2 & \lambda+3 \end{vmatrix} = \begin{vmatrix} \lambda-1 & -2 & 6 \\ 2 & \lambda-5 & 6 \\ 0 & 3-\lambda & \lambda-3 \end{vmatrix} = (\lambda-3)\begin{vmatrix} \lambda-1 & -2 & 6 \\ 2 & \lambda-5 & 6 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= (\lambda-3)\begin{vmatrix} \lambda-1 & 4 & 6 \\ 2 & \lambda+1 & 6 \\ 0 & 0 & 1 \end{vmatrix} = (\lambda-3)\begin{vmatrix} \lambda-1 & 4 \\ 2 & \lambda+1 \end{vmatrix} = (\lambda-3)^2(\lambda+3)$$

So, the characteristic equation is $(\lambda - 3)^2(\lambda + 3) = 0$ and the eigen values are $\lambda = 3, -3$

5. Let
$$A = \begin{pmatrix} -1 & 2 & 2 \\ 4 & 1 & -2 \\ -4 & 2 & 5 \end{pmatrix}$$
   (a) Write down the characteristic equation of A
   (b) Find all the eigenvalues of A.
   (c) For each eigenvalue λ, compute the eigenspace $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

6. Let
$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$
   (a) Write down the characteristic equation of A
   (b) Find all the eigenvalues of A.
   (c) For each eigenvalue λ, compute the eigenspace $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

7. Let
$$A = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$$
   (a) Write down the characteristic equation of A
   (b) Find all the eigenvalues of A.
   (c) For each eigenvalue λ, compute the eigenspace $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

8. Let
$$A = \begin{pmatrix} 4 & 3 & -5 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
   (a) Write down the characteristic equation of A
   (b) Find all the eigenvalues of A.

(c) For each eigenvalue λ, compute the eigenspace $E(\lambda)$, a basis of $E(\lambda)$, and $\dim(E(\lambda))$.

Solution: The characteristic polynomial is $\det(\lambda I - A) =$

$$\begin{vmatrix} \lambda-4 & -3 & 5 & -1 \\ 0 & \lambda+1 & -3 & -1 \\ 0 & 0 & \lambda+1 & 1 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-4)(\lambda+1)^2(\lambda-1)$$

So, the characteristic equation is $(\lambda-4)(\lambda+1)^2(\lambda-1) = 0$ and the eigen values are $\lambda = -1, 1, 4$.

(a) Eigen vectors of $\lambda = -1$ is given by

$$(\lambda I - A)\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5 & -3 & 5 & -1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = t \\ y = -\frac{3}{5}t \\ z = 0 \\ w = 0 \end{cases}$ So, the Eigen space

$$E(-1) = \left\{ \begin{pmatrix} t \\ -\frac{5}{3}t \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Taking $t = 1$, a basis of $E(-1)$ is

$$\left\{ \begin{pmatrix} 1 \\ -\frac{5}{3} \\ 0 \\ 0 \end{pmatrix} \right\} \qquad \text{and} \qquad \dim E(-1) = 1$$

Remark. *Note, although the eigenvalue $\lambda = -1$ has multiplicity two,* $\dim E(-1) = 1$.

(b) Eigen vectors of $\lambda = 1$ is given by

$$(\lambda I - A)\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3 & -3 & 5 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \\ w = t \end{cases}$ So, the Eigen space

$$E(1) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Taking $t = 1$, a basis of $E(1)$ is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ t \end{pmatrix}, \right\} \qquad \text{and} \qquad \dim E(1) = 1$$

(c) Eigen vectors of $\lambda = 1$ is given by

$$(\lambda I - A)\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & -3 & 5 & -1 \\ 0 & 5 & -3 & -1 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving we get $\begin{cases} x = t \\ y = 0 \\ z = 0 \\ w = 0 \end{cases}$ So, the Eigen space

$$E(4) = \left\{ \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Taking $t = 1$, a basis of $E(1)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \right\} \qquad \text{and} \qquad \dim E(4) = 1$$

5.2 Diagonalization

1. **Diagonalize:**
   We did two theorems on dagonalization: Theorem 5.2.2 and Theorem 5.2.3. I will work out one of them.

   (a) Let
   $$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$
   Is A diagonalizable? If yes, write down P such that $P^{-1}AP$ is a diagonal matrix. (Hint:*§5.1, Exercise 2 may be helpful.*)

   Solution: From §5.1, Exercise 2, we know that the characteristic equation is $(\lambda - 2)^2(\lambda - 2) = 0$.

   i. So, $\lambda = 2, 4$ is are the eigenvalues. (**Clue**: *Since $\lambda = 2$ has multiplicity two, we would expect that the $E(2)$ would have dimension two, or less. If not, then A is unlikely to be diagonalizable.*)

   ii. (We are repeating) To compute the eigen space $E(2)$, we solve
   $$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
   $$\begin{cases} -x + z = 0 \\ x - z = 0 \end{cases} \implies \begin{cases} x = z = s \\ y = t \end{cases} \quad \text{where} \quad s, t \in \mathbb{R}.$$
   $$\text{So,} \quad E(2) = \left\{ \begin{pmatrix} s \\ t \\ s \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

Substituting $s = 1, t = 0$ and then $s = 0, t = 1$, a basis of $E(2)$ is given by:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad \text{So,} \qquad \dim E(2) = 2$$

iii. To compute the eigenspace $E(4)$, we solve

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + z = 0 \\ 2y = 0 \end{cases} \implies \begin{cases} x = -z = t \\ y = 0 \end{cases}$$

$$\text{So,} \quad E(4) = \left\{ \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix} : t \in \mathbb{R} \right\}$$

So, a basis of $E(4)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} \qquad \text{So,} \qquad \dim E(4) = 1$$

iv. Finally, $\dim(E(2)) + \dim(E(4)) = 3$. So, we conclude that A is diagonalizable.

v. We form the matrix of the basis eigenvectors.

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}. \quad \text{Then} \quad P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(b) Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Is A diagonalizable? If yes, write down P such that $P^{-1}AP$ is a diagonal matrix. (Hint:*Exercise 1 may be helpful.*)

Solution: For §5.1, Exercise 1, A has two distinct eigenvalues, $\lambda = 0, 2$. Since A has two distinct eigenvalues, we conclude by

Theorem 5.2.3, A is diagonalizable.
Now a basis $E(2)$ is $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.
And a basis of $E(0)$ is $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.
Write

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

(c) Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

Is A diagonalizable? If yes, write down P such that $P^{-1}AP$ is a diagonal matrix. (Hint:*§5.1, Exercise 3 may be helpful.*)

Solution: From §5.1, Exercise 3 , A has two eigenvalues, $\lambda = 0, 2$. A basis for $E(2)$ is

$$\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\} \qquad \text{and} \qquad \dim E(2) = 2$$

A basis of $E(0)$ is

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\} \qquad \text{and} \qquad \dim E(0) = 1$$

Since $\dim E(2) + \dim E(0) = 3$, A is diagonalizable.
Write

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}. \quad \text{Then,} \qquad P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(d) Let

$$A = \begin{pmatrix} 1 & 2 & -6 \\ -2 & 5 & -6 \\ -2 & 2 & -3 \end{pmatrix}$$

Is A diagonalizable? If yes, write down P such that $P^{-1}AP$ is a diagonal matrix. (Hint:*Exercise 4 may be helpful.*)

(e) Let

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 4 & 1 & -2 \\ -4 & 2 & 5 \end{pmatrix}$$

Is A diagonalizable? If yes, write down P such that $P^{-1}AP$ is a diagonal matrix. (Hint:*Exercise 5 may be helpful.*)

(f) Let

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Is A diagonalizable? If yes, write down P such that $P^{-1}AP$ is a diagonal matrix. (Hint:*Exercise 6 may be helpful.*)

(g) Let

$$A = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$$

Is A diagonalizable? If yes, write down P such that $P^{-1}AP$ is a diagonal matrix. (Hint:*Exercise 7 may be helpful.*)

2. **Prove they are not diagonalizable**

(a) Prove that the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable.

Solution: The characteristic polynomial of A is

$$\begin{vmatrix} \lambda - 1 & -3 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2$$

So, A has one eigenvalue $\lambda = 1$. To compute $E(1)$, we solve

$$\begin{pmatrix} 1-1 & -3 \\ 0 & 1-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \qquad x = t, y = 0$$

So,

$$E(1) = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

So, a basis of $E(1)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \qquad \text{And,} \quad \dim E(1) = 1$$

Since $\dim E(1) = 1 \neq 2$, A is not diagonalizable.

(b) Prove that the matrix

$$A = \begin{pmatrix} 1 & 3 & -5 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

is not diagonalizable.

(c) Prove that the matrix

$$A = \begin{pmatrix} 4 & 3 & -5 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is not diagonalizable.

Solution: For §5.1, Exercise 8 , A has three eigenvalues, $\lambda = -1, 1, 4$

Also,

$$\dim E(-1) + \dim E(1) + \dim E(4) = 1 + 1 + 1 = 3 \neq 4$$

So, A is not diagonalizable.

Chapter 6

Inner Product Spaces

6.1 Length and Dot Product

1. **On Length, distance angle, Triangle Inequality** Do not try to simplify your answer too much. You may not get answers in whole numbers.

   (a) Let $\mathbf{u} = (3, 3)$ and $\mathbf{v} = (6, -12)$.

      i. Compute $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u} + \mathbf{v}\|$.

      ii. Compute distance $d(\mathbf{u}, \mathbf{v})$.

      iii. Compute the dot product $\mathbf{u} \cdot \mathbf{v}$.

      iv. Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.
      (*It is okay to leave your answer as* $\cos^{-1}(*)$.)

      v. Verify the Cauchy-Swartz inequality.

      vi. Verify the triangle inequality.

   Solution: We do it one by one:

      i. We have

$$\begin{cases} \|\mathbf{u}\| = \sqrt{3^2 + 3^2} = \sqrt{18}, \\ \|\mathbf{v}\| = \sqrt{6^2 + (-12)^2} = \sqrt{180}, \\ \|\mathbf{u} + \mathbf{v}\| = \|(9, -9)\| = \sqrt{9^2 + (-9)^2} = \sqrt{162} \end{cases}$$

ii. Distance

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(-3,15)\| = \sqrt{(-3)^2+15^2} = \sqrt{234}$$

iii. The dot product

$$\mathbf{u}\cdot\mathbf{v} = 3\cdot 6 + 3\cdot(-12) = -18$$

iv. The angle θ is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-18}{\sqrt{18}\sqrt{180}} = -\frac{1}{\sqrt{10}} \Longrightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{10}}\right)$$

v. To check Cauchy-Swatrz Inequality, we have to check

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|, \quad \text{which works :} \quad |-18| \le \sqrt{18}\sqrt{180}$$

vi. To check triangle inequality, we need to check

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| \quad \text{which works :} \quad \sqrt{162} \le \sqrt{18} + \sqrt{180}$$

(b) Let $\mathbf{u} = (3,3,-3)$ and $\mathbf{v} = (6,6,-12)$.

i. Compute $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u}+\mathbf{v}\|$.
ii. Compute distance $d(\mathbf{u},\mathbf{v})$.
iii. Compute the dot product $\mathbf{u}\cdot\mathbf{v}$.
iv. Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.
(*It is okay to leave your answer as* $\cos^{-1}(*)$.)
v. Verify the Cauchy-Swartz inequality.
vi. Verify the triangle inequality.

(c) Let $\mathbf{u} = (1,1,0)$ and $\mathbf{v} = (\sqrt{6}, \sqrt{6}, -2\sqrt{6})$.

i. Compute $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u}+\mathbf{v}\|$.
ii. Compute distance $d(\mathbf{u},\mathbf{v})$.
iii. Compute the dot product $\mathbf{u}\cdot\mathbf{v}$.
iv. Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.
(*It is okay to leave your answer as* $\cos^{-1}(*)$.)
v. Verify the Cauchy-Swartz inequality.
vi. Verify the triangle inequality.

(d) Let $\mathbf{u} = (1, 1, -1, -1)$ and $\mathbf{v} = (3, 3, 4, 5)$.

i. Compute $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u} + \mathbf{v}\|$.

ii. Compute distance $d(\mathbf{u}, \mathbf{v})$.

iii. Compute the dot product $\mathbf{u} \cdot \mathbf{v}$.

iv. Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.
(*It is okay to leave your answer as* $\cos^{-1}(*)$.)

v. Verify the Cauchy-Swartz inequality.

vi. Verify the triangle inequality.

2. **On Orthogonal Vectors and Pythagorean**

Let me remind the readers that the two words "Orthogonal" and "Perpendicular" means the same thing and used interchangeably.

(a) Let $\mathbf{u} = (1, -1)$ and $\mathbf{v} = (a, a)$. Is $\mathbf{u}$ orthogonal to $\mathbf{v}$. If yes, verify the Pythagorean equality.

(b) Let $\mathbf{u} = (1, -1, 1)$ and $\mathbf{v} = (1, -1, -2)$. Is $\mathbf{u}$ orthogonal to $\mathbf{v}$. If yes, verify the Pythagorean equality.

Solution: We compute Inner (dot) product

$$\mathbf{u} \cdot \mathbf{v} = 1 + 1 - 2 = 0. \quad \text{So, yes} \quad \mathbf{u} \perp \mathbf{v}$$

Now,

$$\begin{cases} \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = (1 + 1 + 1) + (1 + 1 + 4) = 9 \\ \|\mathbf{u} + \mathbf{v}\|^2 = \|(2, -2, -1)\|^2 = 4 + 4 + 1 = 9 \end{cases}$$

So, Pythagorean equality is checked.

(c) Let $\mathbf{u} = (1, -1, 1, -1)$ and $\mathbf{v} = (1, 1, 3, 3)$. Is $\mathbf{u}$ orthogonal to $\mathbf{v}$. If yes, verify the Pythagorean equality.

Solution: We compute Inner (dot) product

$$\mathbf{u} \cdot \mathbf{v} = 1 - 1 + 3 - 3 = 0. \quad \text{So, yes} \quad \mathbf{u} \perp \mathbf{v}$$

Now,

$$\begin{cases} \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = (1 + 1 + 1 + 1) + (1 + 1 + 9 + 9) = 24 \\ \|\mathbf{u} + \mathbf{v}\|^2 = \|(2, 0, 4, 2)\|^2 = 4 + 0 + 16 + 4 = 24 \end{cases}$$

So, Pythagorean equality is checked.

(d) Let $\mathbf{u} = (1, -1, 1, -1)$ and $\mathbf{v} = (a, a, 3a, 3a)$. Is $\mathbf{u}$ orthogonal to $\mathbf{v}$. If yes, verify the Pythagorean equality.

Solution: We compute Inner (dot) product

$$\mathbf{u} \cdot \mathbf{v} = a - a + 3a - 3a = 0. \quad \text{So, yes} \quad \mathbf{u} \perp \mathbf{v}$$

Now,

$$\begin{cases} \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = (1+1+1+1) + (a^2 + a^+ 9a^2 + 9a^2) = 4 + 20a^2 \\ \|\mathbf{u} + \mathbf{v}\|^2 = \|(1+a, -1+a, 1+3a, -1+3a)\|^2 \\ = (1+a)^2 + (-1+a)^2 + (1+3a)^2 + (-1+3a)^2 = 4 + 20a^2 \end{cases}$$

So, Pythagorean equality is checked.

3. **On Computing the Orthogonal Space**

(a) Let $\mathbf{u} = (1, -1, 1)$. Compute the vectors space of all the vectors, orthogonal to $\mathbf{u}$. (*Sometimes, this space is denoted by* $\mathbf{u}^\perp$.)

(b) Let $\mathbf{u} = (1, -1, 1, 3)$. Compute the vectors space of all the vectors, orthogonal to $\mathbf{u}$.

(c) Let $\mathbf{u} = (1, 0, 1, 7)$. Compute the vectors space of all the vectors, orthogonal to $\mathbf{u}$.

Solution: The vectors orthogonal to $\mathbf{u} = (1, 0, 1, 7)$ is given by

$$\mathbf{u} \cdot \mathbf{x} = 0 \Longleftrightarrow x_1 + x_3 + 7x_4 = 0$$

So, the space orthogonal to $\mathbf{u}$ is given by

$$\mathbf{u}^\perp = \left\{ \begin{pmatrix} -t - 7u \\ s \\ t \\ u \end{pmatrix} : s, t, u \in \mathbb{R} \right\}$$

4. **On Changing the direction and size of vectors**

(a) Let $\mathbf{u} = (1, -2, 1)$.

i. Compute the vector $\mathbf{v}$, so the its length $\|\mathbf{v}\| = 1$, and has the same direction.

ii. Compute the vector $\mathbf{v}$, so the its length $\|\mathbf{v}\| = \sqrt{2}$ and has the same direction.

iii. Compute the vector $\mathbf{v}$, so the its length $\|\mathbf{v}\| = \pi$ and has the opposite direction.

(b) Let $\mathbf{u} = (3, -3, 3, 3)$.

i. Compute the vector $\mathbf{v}$, so the its length $\|\mathbf{v}\| = 1$, and has the same direction.

ii. Compute the vector $\mathbf{v}$, so the its length $\|\mathbf{v}\| = \sqrt{2}$ and has the same direction.

iii. Compute the vector $\mathbf{v}$, so the its length $\|\mathbf{v}\| = \pi$ and has the opposite direction.

Solution: We do it one by one:

i. We have

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{(3, -3, 3, 3)}{\sqrt{36}} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

ii. We have

$$\mathbf{v} = \sqrt{2}\frac{\mathbf{u}}{\|\mathbf{u}\|} = \sqrt{2}\frac{(3, -3, 3, 3)}{\sqrt{36}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

iii. We have

$$\mathbf{v} = \pi\frac{\mathbf{u}}{\|\mathbf{u}\|} = \pi\frac{(3, -3, 3, 3)}{\sqrt{36}} = \left(\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$$

6.2 Inner Product Spaces

Remark. *Note that dot product discussed in Section 6.1 is what is called the inner product, in this section. Therefore, $\mathbb{R}^n$, together with dot product, is an inner product space. Other than $\mathbb{R}^n$, bulk of examples on inner product spaces comes from inner product by integration (Example 6.2.2).*

It would not make sense to provide additional problems, on $\mathbb{R}^n$ in this section, just because the name has changed. Most of the problems in this section would on inner product by integration.

1. Suppose $V = C[0, 1]$ be the inner product space of all continuous functions on $[0, 1]$. Let $\mathbf{f}(\mathbf{x}) = e^x$ and $\mathbf{g}(\mathbf{x}) = e^{2x}$.

(a) Compute $\|\mathbf{f}\|$, $\|\mathbf{g}\|$, $\|\mathbf{f}+\mathbf{g}\|$.

(b) Compute distance $d(\mathbf{f},\mathbf{g})$.

(c) Compute the dot product $\mathbf{f}\cdot\mathbf{g}$.

(d) Compute the angle between $\mathbf{f}$ and $\mathbf{g}$.
(*It is okay to leave your answer as* $\cos^{-1}(*)$.)

(e) Verify the triangle inequality.

(f) Compute projection $Proj_{\mathbf{f}}\mathbf{g}$ and $Proj_{\mathbf{g}}\mathbf{f}$.

Solution: We do one by one:

(a) The length

$$\begin{cases} \|\mathbf{f}\| = \sqrt{\int_0^1 e^{2x}dx} = \sqrt{\frac{e^2-1}{2}} \\ \|\mathbf{g}\| = \sqrt{\int_0^1 e^{4x}dx} = \sqrt{\frac{e^4-1}{4}} \\ \|\mathbf{f}+\mathbf{g}\| = \sqrt{\int_0^1 (e^x+e^{2x})^2dx} = \sqrt{\int_0^1 (e^{2x}+2e^{3x}+e^{4x})dx} \\ = \sqrt{\frac{e^2-1}{2} + \frac{2(e^3-1)}{3} + \frac{e^4-1}{4}} \end{cases}$$

(b) The distance $d(\mathbf{f},\mathbf{g}) = \|\mathbf{f}-\mathbf{g}\|$

$$= \sqrt{\int_0^1 (e^x - e^{2x})^2dx} = \sqrt{\int_0^1 (e^{2x} - 2e^{3x} + e^{4x})dx}$$

$$= \sqrt{\frac{e^2-1}{2} - \frac{2(e^3-1)}{3} + \frac{e^4-1}{4}}$$

(c) Now, the inner product

$$\langle \mathbf{f},\mathbf{g}\rangle = \int_0^1 e^{3x}dx = \frac{e^3-1}{3}$$

(d) The angle θ between $\mathbf{f}$ and $\mathbf{g}$ is given by

$$\cos\theta = \frac{\langle \mathbf{f},\mathbf{g}\rangle}{\|\mathbf{f}\|\cdot\|\mathbf{g}\|} = \frac{\frac{e^3-1}{3}}{\sqrt{\frac{e^2-1}{2}}\sqrt{\frac{e^4-1}{4}}}$$

(e) One can use calculator to check, the triangle inequality:

$$\sqrt{\frac{e^2-1}{2}+\frac{2(e^3-1)}{3}+\frac{e^4-1}{4}} \leq \sqrt{\frac{e^2-1}{2}}+\sqrt{\frac{e^4-1}{4}}$$

(f) Finally,

$$\begin{cases} Proj_{\mathbf{f}}\mathbf{g} = \frac{\langle \mathbf{f},\mathbf{g}\rangle}{\|\mathbf{f}\|^2}\mathbf{f} = \frac{\frac{e^3-1}{3}}{\frac{e^2-1}{2}}e^x \\ Proj_{\mathbf{g}}\mathbf{f} = \frac{\langle \mathbf{f},\mathbf{g}\rangle}{\|\mathbf{g}\|^2}\mathbf{g} = \frac{\frac{e^3-1}{3}}{\frac{e^4-1}{4}}e^{2x} \end{cases}$$

2. Suppose $V = C[0,1]$ be the inner product space of all continuous functions on $[0,1]$. Let $\mathbf{f}(\mathbf{x}) = x^3$ and $\mathbf{g}(\mathbf{x}) = 1 - x^3$.

   (a) Compute $\|\mathbf{f}\|$, $\|\mathbf{g}\|$, $\|\mathbf{f}+\mathbf{g}\|$.

   (b) Compute distance $d(\mathbf{f},\mathbf{g})$.

   (c) Compute the dot product $\mathbf{f}\cdot\mathbf{g}$.

   (d) Compute the angle between $\mathbf{f}$ and $\mathbf{g}$.
   (*It is okay to leave your answer as* $\cos^{-1}(*)$.)

   (e) Verify the triangle inequality.

   (f) Compute projection $Proj_{\mathbf{f}}\mathbf{g}$ and $Proj_{\mathbf{g}}\mathbf{f}$.

3. Suppose $V = C[0,\pi]$ be the inner product space of all continuous functions on $[0,1]$. Let $\mathbf{f}(\mathbf{x}) = \cos x$ and $\mathbf{g}(\mathbf{x}) = \sin x$.

   (a) Compute $\|\mathbf{f}\|$, $\|\mathbf{g}\|$, $\|\mathbf{f}+\mathbf{g}\|$.

   (b) Compute distance $d(\mathbf{f},\mathbf{g})$.

   (c) Compute the dot product $\mathbf{f}\cdot\mathbf{g}$.

   (d) Compute the angle between $\mathbf{f}$ and $\mathbf{g}$.
   (*It is okay to leave your answer as* $\cos^{-1}(*)$.)

   (e) Verify the triangle inequality.

   (f) Compute projection $Proj_{\mathbf{f}}\mathbf{g}$ and $Proj_{\mathbf{g}}\mathbf{f}$.

   Hint: Use formulas

$$\begin{cases} \sin 2x = 2\cos x \sin x \\ \cos 2x = \cos^2 x \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \end{cases}$$

6.3 Orthonormal Bases

No Homework Assigned on this section.

Chapter 7

Linear Transformations

7.1 Definitions and Introduction

No Homework Assigned on this section.

7.2 Properties of Linear Transformation

No Homework Assigned on this section.