Matrices: §2.1 Operations with Matrices

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Goals

In this chapter and section we study matrix operations:

- Define **matrix addition**
- Define multiplication of matrix by a scalar, to be called **scalar multiplication**.
- Define multiplication of two matrices, to be called **matrix multiplication**.

Definition

Matrices were defined in §1.2 as an array, with m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \text{Or} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

where a_{ij} are real numbers (for this class).

Size, row and column matrix

- This matrix above is said to have size $m \times n$, because it has m rows and n columns.
- A matrix with equal number of rows and columns is called a **square matrix.** A square matrix of size $n \times n$ is said to have **order** n.
- If a matrix has only one row, it is called a **row matrix**. Likewise, if a matrix has only one column, it is called a **column matrix**.

Notations

1. Often, a matrix is denoted by uppercase letters: $A, B, \ldots$.
2. We also denote the above matrix as $[a_{ij}]$.
3. We may write $A = [a_{ij}]$.

Equality

Two matrices $A = [a_{ij}], B = [b_{ij}]$ are equal, if they have same size $(m \times n)$ and

$$a_{ij} = b_{ij} \qquad for \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

Addition

If $A = [a_{ij}], B = [b_{ij}]$ are two matrices of same size $m \times n$, then their **sum** is defined to be the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

So, the sum is obtained by adding the respective entries.
If the sizes of two matrices are different,
then the sum is NOT defined.

Scalar Multiplication

In this context of matrices of real numbers, by a **scalar** we mean a real number. If $A = [a_{ij}]$ is a $m \times n$ matrix and c is a scalar, then the **scalar multiplication** of A by c is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

Therefore, cA is obtained by multiplying each entry of A by c.

Example of Addition

$$\text{Let} \quad A = \begin{bmatrix} 1 & 1 \\ -3 & 10 \\ 7 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -3 \\ -7 & a \\ b & 3 \end{bmatrix}$$

$$\text{Then} \quad A + B = \begin{bmatrix} 1 & -2 \\ -10 & 10 + a \\ 7 + b & 0 \end{bmatrix},$$

Example of scalar multiplication

Let

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 10 & 7 & -3 \end{bmatrix}$$

Then scalar multiplication by 11 gives

$$11A = \begin{bmatrix} 11 & 11 & -33 \\ 110 & 77 & -33 \end{bmatrix}$$

Matrix Multiplication

Suppose $A = [a_{ij}]$ is a matrix of size $m \times n$ and $B = [b_{ij}]$ is a matrix of size $n \times p$. Then, the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}] \quad where \ \ c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

- **Remarks.** The $(i,j)^{th}$ entry c_{ij} is obtained by "combining" the i^{th} row of A and j^{th} column of B.
- We required that the number of columns of A is equal to the number of rows of B. If they are unequal, then the product AB is NOT defined.

Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix} \quad c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2}$$

Example of matrix multiplication

Let

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 10 & 7 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix},$$

Since number of columns of A and number of rows of B are same, the product AB is defined.

We have

$$AB = \begin{bmatrix} 1*1+1*1+(-3)*2 & 1*1+1*0+(-3)*1 \\ 10*1+7*1+(-3)*2 & 10*1+7*0+(-3)*1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -2 \\ 11 & 7 \end{bmatrix}$$

Remark. Note BA is ALSO defined, which will be a 3×3 matrix. You can compute it similarly.

Matrix form of Linear Systems

A system of linear linear equations can be written in a matrix form: $A\mathbf{x} = \mathbf{b}$ where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Here A is the coefficient matrix.

Example 2.1.1

The solutions of a system can also be written in the matrix form. The system of equations

$$\left\{ \begin{array}{rrrl} 2x & -y & -z & = 0 \\ x & +3y & -z & = 0 \end{array} \right. \quad \text{is same as}$$

$$\left(\begin{array}{ccc} 2 & -1 & -1 \\ 1 & 3 & -1 \end{array} \right) \left(\begin{array}{c} x \\ y \\ z \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right)$$

Continued

Its solutions, can be (*computed and*) written in one of the two ways:

$$x = 4t, \; y = t, \; z = 7t \qquad \text{OR} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}$$

where t is a parameter.

Example 2.1.2

The system from §1.1

$$\left\{\begin{array}{llll} x_1 & & +4x_3 & = 13 \\ 2x_1 & -x_2 & +.5x_3 & = 3.5 \\ 2x_1 & -2x_2 & -7x_3 & = -19 \end{array}\right. \qquad \text{is same as}$$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & -1 & .5 \\ 2 & -2 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 13 \\ 3.5 \\ -19 \end{pmatrix}$$

Continued

Its solution, computed in §1.2 can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 13 \\ 22.5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ -7.5 \\ 1 \end{pmatrix}$$

Example 2.1.3

The system from §1.1

$$\left\{\begin{array}{ccccc} x_1 & & & +3x_4 & = 4 \\ & 6x_2 & -3x_3 & -3x_4 & = 0 \\ & 3x_2 & & -2x_4 & = 1 \\ 2x_1 & -x_2 & +4x_3 & & = 5 \end{array}\right. \quad \text{is same as}$$

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 6 & -3 & -3 \\ 0 & 3 & 0 & -2 \\ 2 & -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 5 \end{pmatrix}$$

Continued

Its solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$