# Matrices: §2.3 The Inverse of Matrices

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# Goals

- Define inverse of a matrix.
- Point out that not every matrix $A$ has an inverse.
- Discuss uniqueness of inverse of a matrix $A$.
- Discuss methods of computing inverses, particularly by row operations.
- Discuss properties of inverses.
- Apply them to solve systems of linear equations.

# Definition:

Let $A$ be a square matrix $A$ (of size $n \times n$).

- $A$ is said to be **invertible** (or **nonsingular**) if there exists a matrix $B$ such that

$$AB = BA = I_n \text{ where } \mathrm{I_n} \text{ is the identity matrix of order } n.$$

- Subsequently, we will see that such a $B$ is unique (if exists), which will be called "the" inverse of $A$.
- Note we assumed that $A$ is a square matrix. We will see, not all square matrices have an inverse.

# Uniqueness of Inverse

**Theorem.** Suppose $A$ is an invetible matrix. Then, its inverse is unique. This unique inverse is denoted by $A^{-1}$.
**Proof.** Since $A$ is invertible, it has at least one inverse. Suppose it has two inverses, $B$ and $C$. By definition

$$AB = BA = I_n = AC = CA. \qquad So,$$

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

So, $B = C$. The proof is complete. ■

# Example: Computing Inverse

**Recall:** In §2.2, HW Problem 3(b), in deed, computes the inverse of the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The same method is elaborated, to compute the inverse of a matrix of order 3, below.

# Inverse of a given Matrix

Let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = .5 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Then $AB = BA = I_2$. So, $A^{-1} = B$.

# Inverse by solving

$$Let \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad and\ let \quad A^{-1} = \begin{bmatrix} a & x & u \\ b & y & v \\ c & z & w \end{bmatrix}$$

Then

$$AA^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & x & u \\ b & y & v \\ c & z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So,

$$\begin{bmatrix} a+b & x+y & u+v \\ a+c & x+z & u+w \\ b+c & y+z & v+w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This gives three systems of linear equations

$$\begin{cases} a+b=1 \\ a+c=0 \\ b+c=0 \end{cases} \begin{cases} x+y=0 \\ x+z=1 \\ y+z=0 \end{cases} \begin{cases} u+v=0 \\ u+w=0 \\ v+w=1 \end{cases}$$

We solve them as before:

$$\begin{cases} a=.5 \\ b=.5 \\ c=-.5 \end{cases} \begin{cases} x=.5 \\ y=-.5 \\ z=.5 \end{cases} \begin{cases} u=-.5 \\ v=.5 \\ w=.5 \end{cases}$$

So,

$$A^{-1} = \begin{bmatrix} a & x & u \\ b & y & v \\ c & z & w \end{bmatrix} = \begin{bmatrix} .5 & .5 & -.5 \\ .5 & -.5 & .5 \\ -.5 & .5 & .5 \end{bmatrix}$$

It is obvious $AA^{-1} = I_n$. We should also check $A^{-1}A = I_n$, which we skip. ■

# An Algorithm to find Inverse

In the above example, we solved three systems of linear equations to find the inverse. An algorithm to do the same by Gauss-Jordan Elimination is as follows:

- let $A$ be a matrix of size $n \times n$.
- Let $I$ be the identity matrix of order $n$
- Form the $n \times 2n$ matrix $[A|I]$ by adjoining $I$ to $A$.
- By row operations,try to reduce $[A|I]$ to the form $[I|B]$. If it works, $A^{-1} = B$; else, $A$ is not invertible.
- Check (ideally) that $AB = BA = I$. (*Subsequently, we will see that this step is not necessary.*)

# Example 2.3.1

We will use the above algorithm to compute inverse of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

We adjoin the identity matrix $I_3$ to $A$, and get

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{bmatrix}$$

Now we apply row operations.

## subtract 3 times first row from second and add first row to third:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & -2 & -5 & 1 & 0 & 1 \end{bmatrix}$$

Subtract 2 times the second row from the first and add 2 times the second row to the last:

$$\begin{bmatrix} 1 & 0 & -4 & 7 & -2 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{bmatrix}$$

## Add 4 times the last row to the first and subtract 3 times the last row from the second:

$$\begin{bmatrix} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{bmatrix}$$

This is in the form $[I_3|B]$. So,

$$A^{-1} = B = \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$$

# Example 2.3.2: non- existence

Here is an example of a matrix, that does not have an inverse.

$$\text{Let} \qquad A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 4 & 8 \\ -2 & 2 & 0 \end{bmatrix}$$

Augment the identity matrix to $A$: $I_3$ to $A$:

$$\begin{bmatrix} 3 & 2 & 5 & 1 & 0 & 0 \\ 4 & 4 & 8 & 0 & 1 & 0 \\ -2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Continued

Switch the last and the first row:

$$\begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 1 \\ 4 & 4 & 8 & 0 & 1 & 0 \\ 3 & 2 & 5 & 1 & 0 & 0 \end{bmatrix}$$

Divide the first row by -2:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -.5 \\ 4 & 4 & 8 & 0 & 1 & 0 \\ 3 & 2 & 5 & 1 & 0 & 0 \end{bmatrix}$$

# Continued

Subtract 4 times the $1^{st}$-row from $2^{nd}$ and subtract 3 times the $1^{st}$ from $3^{rd}$ :

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -.5 \\ 0 & 8 & 8 & 0 & 1 & 2 \\ 0 & 5 & 5 & 1 & 0 & 1.5 \end{bmatrix}$$

Divide second row by 8:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -.5 \\ 0 & 1 & 1 & 0 & .125 & 4 \\ 0 & 5 & 5 & 1 & 0 & 1.5 \end{bmatrix}$$

# Continued

Add $2^{nd}$ row to $1^{st}$ and subtract 5 times $2^{nd}$ row to the last:

$$\begin{bmatrix} 1 & 0 & 1 & .125 & 0 & 3.5 \\ 0 & 1 & 1 & 0 & .125 & 4 \\ 0 & 0 & 0 & 1 & .625 & -18.5 \end{bmatrix}$$

The first half of this matrix does not reduce to the identity $I_3$. So, $A$ does not have an inverse.

# Determinant and Inverse of $2 \times 2$ Matrices

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It follows easily:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

- Define determinant of $A$ as $det(A) = ad - bc$.

- It follows from above,

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad if \quad ad-bc \neq 0.$$

- In next chapter is devoted to determinant of matrices of higher order. It also generalizes this formula for inverse.

# Properties of Inverses

Let $A, B$ be two invertible matrices (of size $n \times n$) , $c \neq 0$ is a scalar and $k$ is a positive integer. Then,

- $\left(A^{-1}\right)^{-1} = A.$
- $\left(A^{k}\right)^{-1} = \left(A^{-1}\right)^{k}.$
- $(cA)^{-1} = \frac{1}{c}A^{-1}$
- $(A^{T})^{-1} = \left(A^{-1}\right)^{T}$
- $(AB)^{-1} = B^{-1}A^{-1}$

# Proof.

In each case, we need to verify the definition of inverse. I will only prove the last one and leave the rest as exercises. We have

$$(AB)\left(B^{-1}A^{-1}\right) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I$$

and similarly $\left(B^{-1}A^{-1}\right)AB = I$. So, the last statement is proved. ■

# Cancellation

Recall, in general, for matrices, $AC = BC$ does not necessarily imply $A = B$. (Please review the example in my notes on §2.2.) But invertible matrices has the cancellation property: Let $C$ be an invertible matrix. Then,

- $AC = BC \implies A = B$.
- $CA = CB \implies A = B$.

# Proof.

Suppose $AC = BC$. On the right side each of this equation, multiply by $C^{-1}$. Then we have,

$$(AC)C^{-1} = (BC)C^{-1} \implies A(CC^{-1}) = B(CC^{-1})$$

$$\Rightarrow A(I) = B(I) \implies A = B.$$

So, the first statement is established. Similiarly, prove the second statement. ■

# Theorem.

Suppose

$$A\mathbf{x} = \mathbf{b} \quad \textit{Systems of Linear Equations}$$

$$\textit{If} \quad A \quad \textit{is invertible}, \ \textit{then} \quad \mathbf{x} = A^{-1}\mathbf{b}.$$

**Proof.**

$$A\mathbf{x} = \mathbf{b} \Rightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow I_n\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$$

The proof is complete. ■

# Example 2.3.3

$$\textit{Compute inverse of} \quad A = \begin{bmatrix} 7 & 3 & -5 \\ -2 & 3 & 2 \\ 3 & 2 & -2 \end{bmatrix}$$

**Solution:** Augment $I_3$ to $A$. We have

$$[A|I_3] = \begin{bmatrix} 7 & 3 & -5 & 1 & 0 & 0 \\ -2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{bmatrix}$$

# Continued

Subtract 2 times the third row from first:

$$\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & -2 \\ -2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{bmatrix}$$

Add 2 times first row to second; then subtract 3 times first row from third:

$$\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 2 & 1 & -4 \\ 0 & 5 & 1 & -3 & 0 & 7 \end{bmatrix}$$

# Continued

Add $2^{nd}$ row to $1^{st}$; then subtract 5 time $2^{nd}$ row from $3^{rd}$:

$$\begin{bmatrix} 1 & 0 & -1 & 3 & 1 & -6 \\ 0 & 1 & 0 & 2 & 1 & -4 \\ 0 & 0 & 1 & -13 & -5 & 27 \end{bmatrix}$$

Add $3^{rd}$ row to $1^{st}$:

$$\begin{bmatrix} 1 & 0 & 0 & -10 & -4 & 21 \\ 0 & 1 & 0 & 2 & 1 & -4 \\ 0 & 0 & 1 & -13 & -5 & 27 \end{bmatrix}$$

# Continued

So,

$$A^{-1} = \begin{pmatrix} -10 & -4 & 21 \\ 2 & 1 & -4 \\ -13 & -5 & 27 \end{pmatrix}$$

# Exercise: Some Algebra

**Exercise 2.3.4**

$$A^{-1}=\begin{pmatrix} 10 & 4 & -21 \\ 2 & 1 & -4 \\ 3 & 1 & -6 \end{pmatrix} \quad B^{-1}=\begin{pmatrix} 4 & 9 & -4 \\ -2 & 4 & -1 \\ 3 & -1 & 5 \end{pmatrix}.$$

- (a) Compute $(AB)^{-1}$
- (b) Compute $(A^T)^{-1}$
- (c) Compute $(2A)^{-1}$

# Solution

**Solution:**

- (a) $(AB)^{-1} = B^{-1}A^{-1} =$

$$\begin{pmatrix} 4 & 9 & -4 \\ -2 & 4 & -1 \\ 3 & -1 & 5 \end{pmatrix} \begin{pmatrix} 10 & 4 & -21 \\ 2 & 1 & -4 \\ 3 & 1 & -6 \end{pmatrix} = \begin{pmatrix} 46 & 21 & -96 \\ -15 & -5 & 32 \\ 43 & 16 & -89 \end{pmatrix}$$

# Solution

**Solution:**

- (b) $(A^T)^{-1} = (A^{-1})^T =$

$$\begin{pmatrix} 10 & 4 & -21 \\ 2 & 1 & -4 \\ 3 & 1 & -6 \end{pmatrix}^T = \begin{pmatrix} 10 & 2 & 3 \\ 4 & 1 & 1 \\ -21 & -4 & -6 \end{pmatrix}$$

# Solution

**Solution:**

- (c) $(2A)^{-1} = \frac{1}{2}A^{-1} =$

$$\frac{1}{2}\begin{pmatrix} 10 & 4 & -21 \\ 2 & 1 & -4 \\ 3 & 1 & -6 \end{pmatrix} = \begin{bmatrix} 5 & 2 & -10.5 \\ 1 & .5 & -2 \\ 1.5 & .5 & -3 \end{bmatrix}$$

# Example: Use inverse to Solve

Use inverse of matrices to solve

$$\begin{cases} 2x_1 & -3x_2 & -5x_3 & = 1 \\ & 3x_2 & -2x_3 & = 3 \\ x_1 & -2x_2 & -2x_3 & = -3 \end{cases}$$

In the matrix form:

$$\begin{pmatrix} 2 & -3 & -5 \\ 0 & 3 & -2 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

# Continued

We write the system as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & -3 & -5 \\ 0 & 3 & -2 \\ 1 & -2 & -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}.$$

- In matrix notation solution is $\mathbf{x} = A^{-1}\mathbf{b}$, if $A^{-1}$ exists.

# Continued

Next, we compute $A^{-1}$. Augment $I_3$ to $A$:

$$[A|I_3] = \left(\begin{array}{cccccc} 2 & -3 & -5 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 0 \\ 1 & -2 & -2 & 0 & 0 & 1 \end{array}\right)$$

Switch first row and last row:

$$\left(\begin{array}{cccccc} 1 & -2 & -2 & 0 & 0 & 1 \\ 0 & 3 & -2 & 0 & 1 & 0 \\ 2 & -3 & -5 & 1 & 0 & 0 \end{array}\right)$$

# Continued

Subtract 2 times $1^{st}$ row from $3^{rd}$:

$$\begin{pmatrix} 1 & -2 & -2 & 0 & 0 & 1 \\ 0 & 3 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -2 \end{pmatrix}$$

Switch $2^{nd}$ and $3^{rd}$ rows:

$$\begin{pmatrix} 1 & -2 & -2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & -2 \\ 0 & 3 & -2 & 0 & 1 & 0 \end{pmatrix}$$

# Continued

Add 2 times $2^{nd}$ row to $1^{st}$ and subtract 3 times $2^{nd}$ from $3^{rd}$:

$$\begin{pmatrix} 1 & 0 & -4 & 2 & 0 & -3 \\ 0 & 1 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 & 1 & 6 \end{pmatrix}$$

Add 4 times $3^{rd}$ row to $1^{st}$ and add $3^{rd}$ row to $2^{nd}$:

$$\begin{pmatrix} 1 & 0 & 0 & -10 & 4 & 21 \\ 0 & 1 & 0 & -2 & 1 & 4 \\ 0 & 0 & 1 & -3 & 1 & 6 \end{pmatrix}$$

# Continued

So, $A^{-1} = \begin{pmatrix} -10 & 4 & 21 \\ -2 & 1 & 4 \\ -3 & 1 & 6 \end{pmatrix}$. And, the solution :

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -10 & 4 & 21 \\ -2 & 1 & 4 \\ -3 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -11 \\ -18 \end{pmatrix}$$

# Another Example

See the old edition of my notes for another example, of such a problem.