Matrices: §2.4 Elementary Matrices

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Goals

- Define Elementary Matrices, corresponding to elementary operations.
- We will see that performing an elementary row operation on a matrix A is same as multiplying A on the left by an elmentary matrix E.
- We will see that any matrix A is invertible if and only if it is the product of elementary matrices.

Definition

Definition: A square matrix A (of size $n \times n$) is called an **Elementary Matrix** if it can be obtained from the identity matrix I_n by a single elementary row operation. That means A is obtained by

- switching two rows on I_n, or
- multiplying a row of I_n by a scalar $c \neq 0$ or
- adding a scalar multiple of a row of I_n to another row.

Inverse of Elementary Matrices

Theorem If E is elementary, then E^{-1} exists and is elementary.

- **Proof** For each of the three types of elementary matrices, write down the inverse and check. I will do it on the board.

Example 2.4.1

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is this matrix elementary. If yes why?
Answer: Yes, it is. The matrix A is obtained from I_3 by adding 3 time the first row of I_3 to the second row.

Example 2.4.2

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

Is this matrix elementary. If yes why?
Answer: Yes, it is. The matrix A is obtained from I_3 by multiplying its third row by 1.5.

Example 2.4.3

Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Is this matrix elementary. If yes why?

Answer: Yes, it is. The matrix A is obtained from I_3 by switching its first and third row.

Theorem 2.1

Theorem. Let A be a matrix of size $m \times n$. Let E be an elementary matrix (of size $m \times m$) obtained by performing an elementary row operation on I_m and B be the matrix obtained from A by performing the same operation on A. Then $B = EA$.

Proof.

We will prove only for one operation (out of three) and when when $n = m = 3$. Suppose E is the matrix obtained by interchanging first and third rows.

$$Then, \quad E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad also\ write \quad A = \begin{bmatrix} x & y & z \\ a & b & c \\ u & v & w \end{bmatrix}$$

$$So, \quad EA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ a & b & c \\ u & v & w \end{bmatrix} = \begin{bmatrix} u & v & w \\ a & b & c \\ x & y & z \end{bmatrix}$$

which is obtained by switching first and third rows of A. ∎

Example 2.4.4

Let

$$A = \begin{bmatrix} 1 & 7 & 1 & 17 \\ -1 & 1 & 1 & 8 \\ 8 & 18 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 18 & 0 & 9 \\ -1 & 1 & 1 & 8 \\ 1 & 7 & 1 & 17 \end{bmatrix}$$

Find an elementary matrix E so that $B = EA$.

Solution: The matrix B is obtained by switching first and the last row of A. They have size 3×4. By the theorem above, E is obtained by switching first and the last row of I_3. So,

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \textit{ so } \quad B = EA \quad \text{(Directly Check, as well.)}.$$

Example 2.4.5

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 8 \\ 8 & 18 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 3 & 10 \\ 8 & 18 & 0 & 9 \end{bmatrix}$$

Find an elementary matrix E so that $B = EA$.

Solution: The matrix B is obtained by adding 2 times the first row of A to the second row of A. By the thorem above, E is obtained from I_3 by adding 2 times its first row to second. So,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; so \quad B = EA \quad \text{(Directly Check, as well.)}.$$

Example 2.4.6

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 8 \\ 8 & 18 & 0 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 9 & 3 & 3 & 24 \\ 8 & 18 & 0 & 9 \end{bmatrix}$$

Find an elementary matrix E so that $B = EA$.

Solution: The matrix B is obtained from A by multiplying its second row by 3. So, by the theorem E is obtained by doing the same to I_3. So

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ so \quad B = EA \quad \text{(Directly Check, as well.)}.$$

Definition

Definition. Two matrices A, B of size $m \times n$ are said to be **row-equivalent** if

$$B = E_k E_{k-1} \cdots E_2 E_1 A \quad \textit{where} \quad E_i \textit{ are elemetary}.$$

This is same as saying that B is obtained from A by application of a series of elemetary row operations.

Theorem 2.2

Theorem. A square matrix A is invertible if and only if it is product of elementary matrices.

Proof. Need to prove two statements. First prove, if A is product it of elementary matrices, then A is invertible. So, suppose $A = E_k E_{k-1} \cdots E_2 E_1$ where E_i are elementary. Since elementary matrices are invertible, E_i^{-1} exists. Write $B = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$. Then

$$AB = (E_k E_{k-1} \cdots E_2 E_1)(E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}) = I.$$

Similarly, $BA = I$. So, B is the inverse of A.

Proof of "only if":

Conversely, assume A is invertible. We have to prove that A is product of elementary matrices. Since A is invertible. The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. So, the augmented matrix $[A|\mathbf{0}]$ reduces to $[I|\mathbf{0}]$ by application of elementary row operations. So, $E_k E_{k-1} \cdots E_2 E_1 [A|\mathbf{0}] = [I|\mathbf{0}]$ where E_i are elementary. So

$$E_k E_{k-1} \cdots E_2 E_1 A = I \quad or \qquad A = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$$

All the factors on the right are elementary. So, A is product of elementary matrices. The proof is complete. ■

Example 2.4.7

Let

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find its inverse, using the theorem above.

Solution. The method is to reduce A to I_3 by elementary operations, and interpret it in terms of multiplication by elementary matrices.

Continued

First, subtract 2 time the first row from second, which is same as multiplying A by the elementary matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad So \quad E_1A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Continued

Now, multiply second row of E_1A by -1. This is same as multiplying E_1A from left by

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad So \quad E_2E_1A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, subtract 3 times the second row from first. So, with

$$E_3 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3E_2E_1A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Continued

Now, multiply the first row of $E_3E_2E_1A$ by .5. So, with

$$E_4 = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

So,

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

If you wish, you can write it more explicitly, by expanding the right hand side.

Example 2.4.8

$$Let \quad A = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 5 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

Find an elementary matrix so that $EA = C$.

Solution. If we add third row of A to its first row, we get C. Let E be the matrix that is obtained from the identity matrix I_3 by adding its third row to the first. Or

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad so \quad EA = C.$$

Example 2.4.9

$$Let \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Compute the inverse of A by elementary operations.

Solution. I_3 is obtained from A by adding -3 times second row of A to third row of A. Accordingly write

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \qquad So, \ EA = I_3, \quad Check \quad AE = I_3.$$

So, $A^{-1} = E$.

Example 2.4.10

$$Let \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 2 & 5 & 7 \end{bmatrix}$$

Find a sequence of elementary matrices whose product is A.

Solution. Let E_1 be the matrix obtained by subtracting the second row of I_3 from its third row and A_1 is obtained by the same operation on A. So,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \quad so \quad E_1A = A_1.$$

E_2 be the the matrix obtained by subtracting 2 times the first row of I_3 from its second row and A_2 is obtained by the same operation on A_1. So,

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad so \quad E_2A_1 = A_2.$$

E_3 be the the matrix obtained by subtracting 2 times the second row of I_3 from its first row and A_3 is obtained by the same operation on A_2. So,

$$E_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad so \quad E_3A_2 = A_3.$$

E_4 be the the matrix obtained by subtracting 3 times the third row of I_3 from its first row and A_4 is obtained by the same operation on A_3. So,

$$E_4 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3, \ \ so\ E_4A_3 = A_4 = I_3.$$

Therefore

$$E_4E_3E_2E_1A = I_3 \quad and \quad A^{-1} = E_4E_3E_2E_4.$$

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_2E_1$$

$$= \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_2E_1 = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_1$$

$$= \begin{bmatrix} 5 & -2 & -3 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_1 = \begin{bmatrix} 5 & -2 & -3 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 & -3 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

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