Determinant: §3.1 The Determinant of a Matrix

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Goals

- We will define determinant of SQUARE matrices, inductively, using the definition of Minors and cofactors.
- We will see that determinant of triangular matrices is the product of its diagonal elements.
- Determinants are useful to compute the inverse of a matrix and solve linear systems of equations (Cramer's rule).

Overview of the definition

- Given a square matrix A, the determinant of A will be defined as a number, to be denoted by $det(A)$ or $|A|$.
- Given such a matrix A of size $n \times n$, it is possible to give a direct definition (or a formula) of $det(A)$. Unfortunately, this may be beyond the scope at this level.
- Therefore, we define inductively. That means, we first define determinant of 1×1 and 2×2 matrices. Use this to define determinant of 3×3 matrices. Then, use this to define determinant of 4×4 matrices and so.

Determinant of 1×1 and 2×2 matrices

- For a 1×1 matrix $A = [a]$ define $det(A) = |A| = a$.
- Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad define \quad det(A) = |A| = ad - bc.$$

Example 3.1.1

Let

$$A = \begin{bmatrix} 2 & 17 \\ 3 & -2 \end{bmatrix} \quad then \quad det(A) = |A| = 2*(-2)-17*3 = -53$$

Example 3.1.2

Let

$$A = \begin{bmatrix} 3 & 27 \\ 1 & 9 \end{bmatrix} \quad then \quad det(A) = |A| = 3 * 9 - 1 * 27 = 0.$$

Minors of 3 × 3 matrices

First, we define Minors and Cofactors of 3 × 3 matrices.
Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then, the **Minor** M_{ij} of a_{ij} is defined to be the determinant of the 2×2 matrix obtained by deleting the i^{th} row and j^{th} column.

For example

$$M_{22} = \begin{vmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Like wise

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

Cofactors of 3×3 matrices

Let A the 3×3 matrix as in the above frame. Then, the **Cofactor** C_{ij} of a_{ij} is defined, by some sign adjustment of the minors, as follows:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, using the above frame

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = a_{22}a_{33} - a_{23}a_{33}$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -(a_{11}a_{32} - a_{12}a_{31})$$

$$C_{32} = (-1)^{3+2} M_{32} = -(a_{11}a_{23} - a_{13}a_{21}).$$

Determinant of 3×3 matrices

Let A be the 3×3 matrix as above. Then the **determinant** of A is defined by

$$\det(A) = |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This definition may be called "definition by expansion by cofactors, along the first row". It is possible to define the same by expansion by second of third row, which we will be discussed later.

Example 3.1.3

Let

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 0 \\ -2 & 1 & 1 \end{vmatrix}$$

Compute the minors M_{11}, M_{12}, M_{13}, the cofactors C_{11}, C_{12}, C_{13} and the determinant of A.

Solution:

Then minors

$$M_{11} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}, M_{12} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix}, M_{13} = \begin{vmatrix} 3 & -2 \\ -2 & 1 \end{vmatrix}$$

Or

$$M_{11} = -2, \quad M_{12} = 3, \quad M_{13} = -1$$

Continued

So, the cofactors

$$C_{11} = (-1)^{1+1}M_{11} = -2, \quad C_{12} = (-1)^{1+2}M_{12} = -3,$$

$$C_{13} = (-1)^{1+3}M_{13} = -1$$

So,

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2*(-2) + 1*(-3) + 1*(-1) = -8$$

Example 3.1.4

Let

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 0 & 5 \end{vmatrix}$$

Compute the determinant of A.

Solution:

Then minors

$$M_{11} = \begin{vmatrix} -1 & 1 \\ 0 & 5 \end{vmatrix}, M_{12} = \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}, M_{13} = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$$

Or

$$M_{11} = -5, \quad M_{12} = 6, \quad M_{13} = 4$$

Continued

So, the cofactors

$$C_{11} = (-1)^{1+1}M_{11} = -5, \quad C_{12} = (-1)^{1+2}M_{12} = -6,$$

$$C_{13} = (-1)^{1+3}M_{13} = 4$$

So,

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2*(-5) + 1*(-6) + 1*4 = -12$$

Example 3.1.5

Let

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 0 \end{vmatrix}$$

Compute the determinant of A.

Solution:

Then minors

$$M_{11} = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix}, M_{12} = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}, M_{13} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

Or

$$M_{11} = 3, \quad M_{12} = 2, \quad M_{13} = -1$$

Continued

So, the cofactors

$$C_{11} = (-1)^{1+1} M_{11} = 3, \quad C_{12} = (-1)^{1+2} M_{12} = -2,$$

$$C_{13} = (-1)^{1+3} M_{13} = -1$$

So,

$$|A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = 1 * 3 + 1 * (-2) + 1 * (-1) = 0$$

Alternative Method for 3×3 matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Form the 3×5 matrix by augmenting $1^{st}, 2^{end}$ columns to A:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & 31 & a_{32} \end{matrix}$$

Continued

Then $|A|$ can be computed as follows:

- add the product of all three entries in the three left to right diagonal
- add the product of all three entries in the three right to left diagonal
- Then, $|A|$ is the difference.

Exercise: Compute the determinant of the matrices in Example 1.3.3-5, using this method.

Inductive process of definition

- We defined determinant of matrices size 3×3, using the determinant of 2×2 matrices.
- Now, we do the same for 4×4 matrices. This means first define minors, which would be determinant of 3×3 matrices. Then, define Cofactors by adjusting the sign of the Minors.Then, use the cofactors of define the determiant of the 4×4 matrix.
- Then, we can define minors, cofactors and determinant of 5×5 matrices. The process continues.

Minors of $n \times n$ Matrices

We assume that we know how to define determiant of $(n-1) \times (n-1)$ matrices. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

be a square matrix of size $n \times n$. The **minor** M_{ij} of a_{ij} is defined to be the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column.

Cofactors and Determinant of $n \times n$ Matrices

Let A be a $n \times n$ matrix.

- Define

$$C_{ij} = (-1)^{i+j} M_{ij} \quad \textit{which iscalled the } \textbf{cofactor} \textit{ of } a_{ij}.$$

- Define

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

This would be called a definiton by expasion by cofactors, along first row.

Theorem 3.1.1

Theorem 3.1.1 Let A be an $n \times n$ matrix as above. Then,

- $|A|$ can be computed by expanding by any row (i^{th} row):

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

- $|A|$ can ALSO be computed by expanding by any column (j^{th} column):

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

- **Proof.** Proof is needed, which we skip.

The general definition, without using induction

This may be beyond the scope at this level. However, I will attempt to explain. Let A be a $n \times n$ matrix, as above.

- **Permutation:** Any rearrangement σ of the set $\{1, 2, \ldots, n\}$ is called a permutation of $\{1, 2, \ldots, n\}$. So, σ would look like $\sigma = i_1, i_2, \ldots, i_n$ where $i_j \in \{1, 2, \ldots, n\}$ and each one appears only once.
- To each such premutation σ a sign (signature) ± 1 is attached, which we will not explain. $sign(\sigma) = 1 \ or \ -1$.

- We have

$$|A| = \sum_{all\ \sigma} sign(\sigma) a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n}$$

where $\sigma = j_1, j_2, \ldots, j_n$ and $\pm$ is the sign of σ.

- In fact, we take products of n entries, such that exactly one factor comes from each row and each column, then adjust the sign of such products and add.

Definition.

Definitions. Let A be a $n \times n$ matrix.

- We say A is Upper Triangular matrix, all entries of A below the main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i > j$.
- We say A is Lower Triangular matrix, all entries of A above he main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i < j$.

Theorem 3.1.2

Theorem 3.1.2 Let A be a triangular matrix of order n. Then $|A|$ is product of the main-diagonal entries. Notationally,

$$|A| = a_{11}a_{22}\cdots a_{nn}.$$

Proof. The proof is easy when $n = 1, 2$. We prove it when $n = 3$. Let use assume A is lower triangular. So,

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Continued

We expand by the first row:

$$|A| = a_{11}C_{11} + 0C_{12} + 0C_{13} = a_{11}C_{11}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

For upper triangular matrices, we can prove similarly, by column expansion. For higher order matrices, we can use mathematical induction. ■

Example 1.3.6

Compute the determinant, by expansion by cofactors, of

$$A = \begin{bmatrix} x & y & z \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Solution.

- The cofactors

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 4 \\ 0 & 2 \end{vmatrix} = 8, \; C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = 2$$

- $$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} = -4$$
- So, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$$= x * 8 + y * 2 + z * (-4) = 8x + 2y - 4z$$

Example 3.1.7.

$$Let \quad A = \begin{bmatrix} 3 & 7 & -3 & 13 \\ 0 & -7 & 2 & 17 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad Compute\ det(A).$$

Solution. This is an upper triangular matrix. So, $|A|$ is the product of the diagonal entries. So

$$|A| = 3 * (-7) * 4 * 5 = -420.$$

Example 3.1.8.

$$Let \quad A = \begin{bmatrix} 3 & 7 & -3 & 13 \\ 0 & -7 & 2 & 17 \\ 3 & 7 & 1 & 16 \\ -3 & -7 & 3 & -8 \end{bmatrix} \quad Compute\ det(A).$$

Solution. We expand, by first row. First, we compute the minors, and cofactors, of the elements in the first row.

$$M_{11} = \begin{vmatrix} -7 & 2 & 17 \\ 7 & 1 & 16 \\ -7 & 3 & -8 \end{vmatrix} = 756 \quad C_{11} = (-1)^{1+1} M_{11} = 756$$

Continued

$$M_{12} = \begin{vmatrix} 0 & 2 & 17 \\ 3 & 1 & 16 \\ -3 & 3 & -8 \end{vmatrix} = 156 \quad C_{12} = (-1)^{1+2} M_{12} = -156$$

$$M_{13} = \begin{vmatrix} 0 & -7 & 17 \\ 3 & 7 & 16 \\ -3 & -7 & -8 \end{vmatrix} = 168 \quad C_{13} = (-1)^{1+3} M_{13} = 168$$

$$M_{14} = \begin{vmatrix} 0 & -7 & 2 \\ 3 & 7 & 1 \\ -3 & -7 & 3 \end{vmatrix} = 84 \quad C_{14} = (-1)^{1+4} M_{14} = -84$$

Continued

So,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$
$$= 3 * 756 + 7 * (-156) - 3 * 168 + 13 * (-84) = -420$$

Example 3.1.9

$$Let \quad A = \begin{bmatrix} 3 & 7 & 1 & 21 \\ 3 & 0 & 3 & 38 \\ 3 & 7 & 1 & 16 \\ -3 & -7 & 3 & -8 \end{bmatrix} \quad Compute\ det(A).$$

Solution. We expand, by first row. First, we compute the minors, and cofactors, of the elements in the first row.

$$M_{11} = \begin{vmatrix} 0 & 3 & 38 \\ 7 & 1 & 16 \\ -7 & 3 & -8 \end{vmatrix} = 896 \quad C_{11} = (-1)^{1+1} M_{11} = 896$$

Continued

$$M_{12} = \begin{vmatrix} 3 & 3 & 38 \\ 3 & 1 & 16 \\ -3 & 3 & -8 \end{vmatrix} = 216 \quad C_{12} = (-1)^{1+2} M_{12} = -216$$

$$M_{13} = \begin{vmatrix} 3 & 0 & 38 \\ 3 & 7 & 16 \\ -3 & -7 & -8 \end{vmatrix} = 168 \quad C_{13} = (-1)^{1+3} M_{13} = 168$$

$$M_{14} = \begin{vmatrix} 3 & 0 & 3 \\ 3 & 7 & 1 \\ -3 & -7 & 3 \end{vmatrix} = 84 \quad C_{14} = (-1)^{1+4} M_{14} = -84$$

Continued

So,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$= 3 * 896 + 7 * (-216) + 1 * 168 + 21 * (-84) = -420$$

Example 3.1.10

$$\textit{Solve} \quad \begin{vmatrix} x+1 & 1 \\ -1 & x-1 \end{vmatrix} = 0$$

Solution. So,

$$(x+1)(x-1)+1=0 \quad or \quad x^2=0$$

So, $x=0$.

Example 3.1.11

$$Solve \quad \begin{vmatrix} x^2+1 & 2 \\ x & 1 \end{vmatrix} = 0$$

Solution. So,

$$(x^2+1)*1 - 2x = 0 \quad or \quad (x-1)^2 = 0$$

So, $x = 1$.