Determinate: §3.2 Evaluation of Determinant with Elementary Operations

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"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." - Albert Einstein
Goals

- Discuss effect of elementary operations on determinants.
- Use them to compute determinant of a matrix $A$ by reducing it to a simpler matrix (like triangular matrices).
- This method is helpful, because expansion by cofactors may take too long.
Effect of Elementary Operations

Theorem 3.2.3. Let $A, B$ be two square matrices of same size.

- If $B$ is obtained by interchanging two rows of $A$, then
  \[ |B| = -|A| \]

- If $B$ is obtained by adding a scalar multiple of a row of $A$ to another row of $A$, then
  \[ |B| = |A| \]

- If $B$ is obtained by multiplying a row of $A$ by a scalar $c$, then
  \[ |B| = c|A| \]
Elementary Column Operations

- Like elementary row operations, there are three elementary column operations: Interchanging two columns, multiplying a column by a scalar $c$, and adding a scalar multiple of a column to another column.

- Two matrices $A, B$ are called column-equivalent, if $B$ is obtained by application of a series of elementary column operations to $A$.

- Theorem 3.3 remains valid if the word ”row” is replaced by ”column”.
Suppose $A$ is a square matrix. Assume one of the following three holds:

- An entire row (or column) of $A$ is zero, OR
- two rows (or columns) are equal, OR
- one row (or column) is a scalar multiple of another row (or column).

Then, $|A| = 0$. 
Example 3.2.1

Use elementary operations to compute the determinant of

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
3 & 0 & -1 \\
1 & -2 & -1 \\
\end{bmatrix}
\]

Idea is to **reduce it to a triangular matrix** by elementary row and column operations. Subtract 3 times first row from second:

\[
|A| = \begin{vmatrix}
1 & 1 & 1 \\
0 & -3 & -4 \\
1 & -2 & -1 \\
\end{vmatrix}
\]
Subtract first row from the third and then take out $-1$ from the second row:

\[
|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & -3 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & -3 & -2 \end{vmatrix}
\]

Add second row to third:

\[
|A| = - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{vmatrix} = -1 \times 3 \times 2 = -6
\]
Example 3.2.2

Use elementary operations to compute the determinant of

\[ A = \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 3 & -7 & 13 \end{bmatrix} \]

We try to reduce it to a triangular matrix. Subtract the first row from last:

\[ A = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & -15 & 20 \end{vmatrix} \]
Subtract 3 times second row from third:

\[ A = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & 0 & 8 \end{vmatrix} = 3 \cdot (-5) \cdot 8 = -120 \]
Example 3.2.3

Use elementary operations to compute the determinant of

\[ A = \begin{bmatrix} 0 & -3 & 8 & 2 \\ 15 & 1 & -1 & -8 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{bmatrix} \]

We will try to reduce it to a triangular matrix. Add 2 times fourth row to the second and then switch first and second rows.

\[ |A| = \begin{vmatrix} 0 & -3 & 8 & 2 \\ 1 & 1 & -1 & 20 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & -3 & 8 & 2 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix} \]
Now add 4 times the first row to the third and then add 7 times the first row to the fourth:

\[
|A| = - \begin{vmatrix}
1 & 1 & -1 & 20 \\
0 & -3 & 8 & 2 \\
0 & 10 & -4 & 89 \\
0 & 1 & -1 & 22 \\
\end{vmatrix}
\]

Take out 7 from fourth row and then switch second and fourth row:

\[
|A| = -7 \begin{vmatrix}
1 & 1 & -1 & 20 \\
0 & -3 & 8 & 2 \\
0 & 10 & -4 & 89 \\
0 & 1 & -1 & 22 \\
\end{vmatrix} = +7 \begin{vmatrix}
1 & 1 & -1 & 20 \\
0 & 1 & -1 & 22 \\
0 & 10 & -4 & 89 \\
0 & -3 & 8 & 2 \\
\end{vmatrix}
\]
Subtract 10 times the second row from third, and then add 3 time the second row to fourth:

$$|A| = 7 \begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 6 & -131 \\ 0 & -3 & 8 & 2 \end{vmatrix} = 7 \begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 6 & -131 \\ 0 & 0 & 5 & 68 \end{vmatrix}$$

Now subtract fourth row from third and then subtract 5 times the third row from fourth:

$$|A| = 7 \begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -199 \\ 0 & 0 & 5 & 68 \end{vmatrix} = 7 \begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -199 \\ 0 & 0 & 0 & 1063 \end{vmatrix}$$
So,

\[ |A| = 7(1 \times 1 \times 1 \times 1063) = 7441 \]