Determinant: §3.2 Evaluation of Determinant with Elementary Operations

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"*As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*" - Albert Einstein

Goals

- Discuss effect of elementary operations on determinants.
- Use them to compute determinant of a matrix A by reducing it to a simpler matrix (like triangular matrices).
- This method is helpful, because expansion by cofactors may take too long.

Effect of Elementary Operations

Theorem 3.2.3. Let A, B be two square matrices of same size.

- If B is obtained by interchanging two rows of A, then
$$|B| = -|A|$$
- If B is obtained by adding a scalar multiple of a row of A to another row of A, then
$$|B| = |A|$$
- If B is obtained by multiplying a row of A by a scalar c, then
$$|B| = c|A|$$

Elementary Column Operations

- Like elementary row operations, there are three elementary column operations: Interchanging two columns, multiplying a column by a scalar c, and adding a scalar multiple of a column to another column.
- Two matrices A, B are called column-equivalent, if B is obtained by application of a series of elementary column operations to A.
- Theorem 3.3 remains valid if the word "row" is replaced by "column".

Zero Determinant

Suppose A is a square matrix. Assume one of the following three holds:

- An entire row (or column) of A is zero, OR
- two rows (or columns) are equal, OR
- one row (or column) is a scalar multiple of another row (or column).

$$Then, \qquad |A| = 0.$$

Example 3.2.1

Use elementary operations to compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & -1 \\ 1 & -2 & -1 \end{bmatrix}$$

Idea is to reduce it to a triangular matrix by elementary row and column operations. Subtract 3 times first row from second:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 1 & -2 & -1 \end{vmatrix}$$

Continued

Subtract first row from the third and then take out -1 from the second row:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & -3 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & -3 & -2 \end{vmatrix}$$

Add second row to third:

$$|A| = - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{vmatrix} = -1 * 3 * 2 = -6$$

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Example 3.2.2

Use elementary operations to compute the determinant of

$$A = \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 3 & -7 & 13 \end{bmatrix}$$

We try to reduce it to a triangular matrix. Subtract the first row from last:

$$|A| = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & -15 & 20 \end{vmatrix}$$

Continued

Subtract 3 times second row from third:

$$|A| = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & 0 & 8 \end{vmatrix} = 3 * (-5) * 8 = -120$$

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Example 3.2.3

Use elementary operations to compute the determinant of

$$A = \begin{bmatrix} 0 & -3 & 8 & 2 \\ 15 & 1 & -1 & -8 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{bmatrix}$$

We will try to reduce it to a triangular matrix. Add 2 times fourth row to the second and then switch first and second rows.

$$|A| = \begin{vmatrix} 0 & -3 & 8 & 2 \\ 1 & 1 & -1 & 20 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & -3 & 8 & 2 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$$

Continued

Now add 4 times the first row to the third and then add 7 times the first row to the fourth:

$$|A| = -\begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & -3 & 8 & 2 \\ 0 & 10 & -4 & 89 \\ 0 & 7 & -7 & 154 \end{vmatrix}$$

Take out 7 from fouth row and then switch second and fourth roe:

$$|A| = -7\begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & -3 & 8 & 2 \\ 0 & 10 & -4 & 89 \\ 0 & 1 & -1 & 22 \end{vmatrix} = +7\begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 10 & -4 & 89 \\ 0 & -3 & 8 & 2 \end{vmatrix}$$

Continued

Subtract 10 times the second row from third, and then add 3 time the second row to fourth:

$$|A| = 7\begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 6 & -131 \\ 0 & -3 & 8 & 2 \end{vmatrix} = 7\begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 6 & -131 \\ 0 & 0 & 5 & 68 \end{vmatrix}$$

Now subtract fourth row from third and then subtract 5 times the third row from fourth:

$$|A| = 7\begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -199 \\ 0 & 0 & 5 & 68 \end{vmatrix} = 7\begin{vmatrix} 1 & 1 & -1 & 20 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -199 \\ 0 & 0 & 0 & 1063 \end{vmatrix}$$

Continued

So,

$$|A| = 7(1 * 1 * 1 * 1063) = 7441$$