Vector Spaces
§4.1 Vectors in $n-$Spaces

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Goals

- Get familar with real $n-$spaces $\mathbb{R}^n$, like $\mathbb{R}^2$, $\mathbb{R}^3$.
- Get familiar with some of the properties of the real $n-$spaces $\mathbb{R}^n$.
- Justify why points in $\mathbb{R}^n$ can be viewed as "vectors" as in physics and engineering.

We will be brief for this section.

The real $n-$ spaces $\mathbb{R}^n$

- The set of all real numbers is denoted by $\mathbb{R}$. The real line is the geometric realization of $\mathbb{R}$.
- The set of all ordered pairs (x, y) of real numbers is denoted by $\mathbb{R}^2$. The real plane is the geometric realization of $\mathbb{R}^2$.
- The set of all ordered triples (x, y, z) of real numbers is denoted by $\mathbb{R}^3$. The 3-space is the geometric realization of $\mathbb{R}^3$.

Definition: $n-$ spaces $\mathbb{R}^n$

Definition. For any integer n, set of all **ordered** $n-$**tuples** $(x_1, x_2, \ldots, x_n)$ of real numbers is denoted by $\mathbb{R}^n$. $\mathbb{R}^n$ is called the (real) $n-$**space.**

- Elements $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ are called points in $\mathbb{R}^n$. They are also, sometimes, called **vectors** in $\mathbb{R}^n$.

Properties of the $n-$space $\mathbb{R}^n$

The $n-$space $\mathbb{R}^n$ enjoys many properties of the plane $\mathbb{R}^2$ and the $3-$space $\mathbb{R}^3$. Like row matrices, addition and scalar multiplication are defined on $\mathbb{R}^n$. For points $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and scalar $c \in \mathbb{R}$

- define addition

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

- and define scalar multiplication

$$c\mathbf{x} = (cx_1, cx_2, \ldots, cx_n).$$

- The negative of $\mathbf{x}$ is defined as

$$-\mathbf{x} = (-x_1, -x_2, \ldots, -x_n).$$

Properties: Theorem 4.1.1

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are three vectors in $\mathbb{R}^n$ and $c, d \in \mathbb{R}$ two scalars. Then,

- (1. Closure under addition): $\mathbf{u} + \mathbf{v}$ is a vector in $\mathbb{R}^n$. (We say: $\mathbb{R}^n$ *is closed under addition*.)
- (2. Commutativity): $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- (3. Associativity I): $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- (4. Additive Identity or zero): $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
- (5. Additive inverse): $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Continued:

- (6. Closure under scalar multiplication): $c\mathbf{u}$ is a vector in $\mathbb{R}^n$. (We say: $\mathbb{R}^n$ *is closed under scalar multiplication*.)
- (7. Distributivity I): $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
- (8. Distributivity II):$(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- (9. Associativity II): $c(d\mathbf{u}) = (cd)\mathbf{u}$
- (10. Multiplicative Identity): $1\mathbf{u} = \mathbf{u}$

Remark. These properties will motivate and be used to give an "abstract" definition of Vector Spaces in §4.2. This is the main (or only) purpose of this section.

Notations:

Notations: A vector $\mathbf{x}$ in the $n-$space $\mathbb{R}^n$ is denoted by all the following ways:

- As a row matrix

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \quad or \quad \mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

- as a column matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \quad or \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

Why call them vectors?

In physics and engineering, a vector **u** is defined by the following two properties:

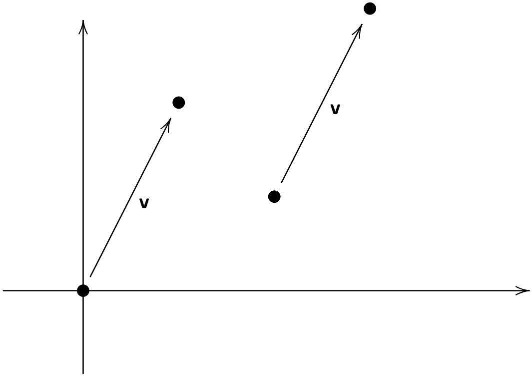
- **u** has a magnitude ℓ (or norm $||\mathbf{u}||$) and
- **u** has a direction θ.

Example: Among the examples of vectors are: *velocity, acceleration, force*. On the other hand, *speed* is not a vector because it has a magnitude, but not a direction.

Arrows as vectors

Vectors can be represented by arrows in the plane or three dimensional space. For example, for dynamics in the plane, force, velocity, accellaration can be represented by an arrow (vector):
For example, velocity $\mathbf{v}$ can be represented by arrows in the $\mathbb{R}^3$: where

- the length of the arrow represnts the magniture of $\mathbf{v}$
- the direction is shown by the direction of the arrow.



- Parallel arrows of same length represent the same vector **v**. Each representation has an initial point and a terminal point, indicated by the •.

Continued

- A vector **v** is described by magnitude ℓ and direction θ. So, it can be represented by ANY arrow of length ℓ so that the angle between $x-$axis and **v** is θ. So, any two arrows of that are parallel and of same length, represent the same vector **v**.
- However, there is one unique such arrow that starts at the origin. So, the vectors are in one to one correspondance with the arrows starting at the origin.
- An arrow starting at the origin is determined by its terminal point (x, y). So, the arrows starting at the origin are in one to one correspondance with the points (x, y) in the plane $\mathbb{R}^2$.

Continued

- This establishes that vectors in plane are in one to one correspondance with the points (x, y) in the plane $\mathbb{R}^2$.
- This is the reason why points is $\mathbb{R}^2$ (more generally in $\mathbb{R}^n$) are, sometimes, called vectors. Further, the $n-$space $\mathbb{R}^n$ would be the prime examples of so called **Vector Spaces** in §4.2.