Vector Spaces
§4.2 Vector Spaces

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Goals

- Give definition of Vector Spaces
- Give examples and non-examples of Vector Spaces

Operations on Sets

- On the set of integers $\mathbb{Z}$, or on the set of real numbers $\mathbb{R}$ we worked with addition $+$, multiplication $\times$.
- On the $n-$space $\mathbb{R}^n$, we have addition and scalar multiplication.
- These are called operations on the respective sets. Such an operation associates an oredered pair to an element in V, like,

$$(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v}, \quad or \quad (c, \mathbf{v}) \mapsto c\mathbf{v}$$

These are called binary operations, because they associate an ordered pair to an element in V.

Operations on Sets: Continued

- Likewise, in mathematics, we define the same on any set V. Given two sets V, R, a binary operation $*, o$ associates an ordered pair to an element in V.

$$* : V \times V \longrightarrow V \quad (u, v) \mapsto u * v$$

$$o : R \times V \longrightarrow V \quad (c, v) \mapsto cov$$

  Such operations are mostly denoted by $+, \times$ and called addition, multiplication or scalar multiplication, depending on the context.
- Examples of Binary operations include:(a) Matrix addition, multiplication; (b) polynomial addition and multiplication;(c) addition, multiplication and composition of functions.

Vector Spaces: Motivations

There are many mathematical sets V, with an addition $+$ and a scalar multiplication, satisfy the properties of the vectors in $n-$spaces $\mathbb{R}^n$, listed in Theorem 4.1 and 4.2. Examples of such sets include

- the set all of matrices $M_{m,n}$ of size $m \times n$,
- set of polynomial,
- Set of all real valued continuous functions on a set.

In order to unify the study of such sets, abstract vector spaces are defined, simply by listing the properties in Theorem 4.1.1 in §4.1.

Vector Spaces: Definition

Definition: Let V be a set with two operations (**vector addition** $+$ and scalar multiplication). We say that V is a Vector Space over the real numbers $\mathbb{R}$ if, for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars (reals) c, d, the following properties are stistified:

- (1. Closure under addition): $\mathbf{u} + \mathbf{v}$ is in V.
- (2. Commutativity): $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- (3. Associativity I): $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- (4. Additive Identity or zero):There is an element in V, denoted by $\mathbf{0}$ and to be called a (the) zero vector such that

$$\mathbf{u} + \mathbf{0} = \mathbf{u}, \quad \textit{for every} \quad \mathbf{u} \in V.$$

Continued:

- (5. Additive inverse): For every $\mathbf{u} \in V$ there is and an element in $\mathbf{x} \in V$, denoted $\mathbf{x}$ such that $\mathbf{u} + \mathbf{x} = \mathbf{0}$. Such an $\mathbf{x}$ would be called the/an additive inverse of $\mathbf{u}$.
- (6. Closure under scalar multiplication): $c\mathbf{u}$ is in V
- (7. Distributivity I): $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
- (8. Distributivity II):$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- (9. Associativity II): $c(d\mathbf{u}) = (cd)\mathbf{u}$
- (10. Multiplicative Identity): $1\mathbf{u} = \mathbf{u}$

Four Entities of Vector Spaces

Note that a Vector Space as four entities:

- (1) A set of vectors V,
- (2) a set of scalars,
- (3,4) two operations.
- In this course, the set of scalars is $\mathbb{R}$. The theory of Vector spaces over complex scalars $\mathbb{C}$ would be exactly analogous, while we avoid it in this course.

Standard Examples

We give a list of easy examples.

- **Example 1:** The plane, $\mathbb{R}^2, \mathbb{R}^3$ with standard addition and scalar multiplication is a Vector Space. More generally, the $n-$space, $\mathbb{R}^n$ with standard addition and scalar multiplication is a Vector Space.
- **Example 2:** Let $M_{2,4}$ be the set of all 2×4. So,

$$M_{2,4} = \left\{ \begin{pmatrix} a & b & c & d \\ x & y & z & w \end{pmatrix} : a, b, c, d, x, y, z, w \in \mathbb{R} \right\}$$

Then, with standard addition and scalar multiplication $M_{2,4}$ is a Vector Space.

Examples from the Textbook

- **Example 3:** Let P_n denote the set of all polynomials of degree less or equal to n. So,

  $$P_2 = \{a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0 : \ a_i \in \mathbb{R} \ \forall \ i = 0, 1, \ldots, n\}$$

  With standard addition and scalar multiplication P_n is a Vector Space.
- **Example 4.** Let I be an interval and $C(I)$ denotes the set of all real-valued continuous functions on I. Then, with standard addition and scalar multiplication $C(I)$ is a Vector Space. For example,

  $$C(-\infty, \infty), C(0,1), C[0,1], C(0,1], C[1,0)$$

  are vector spaces.

Formal Proofs

To give a proof we need to check all the 10 properties in the definition. While each step may be easy, students at this level are not used to writing a formal proof. Here is a proof that $C(0,1)$ is a vector space.

- So, the vectors are continuous functions $\mathbf{f}(x) : (0,1) \longrightarrow \mathbb{R}$.
- For vectors $\mathbf{f}, \mathbf{g} \in C(0,1)$ addition is defined as follows:

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$$

- For $\mathbf{f} \in C(0,1), c \in \mathbb{R}$ scalar multiplication is defined as

$$(c\mathbf{f})(x) = c(\mathbf{f}(x)).$$

Formal Proofs: Continued

For this addition and scalar multiplication, we have to check all 10 properties in the defintion. Suppose $\mathbf{f}, \mathbf{g}, \mathbf{h} \in C(0,1)$ and $c, d \in \mathbb{R}$.

- (1. Closure under addition): $C(0,1)$ is closed under addition. This is because sum $\mathbf{f} + \mathbf{g}$ of two continuous functions $\mathbf{f}, \mathbf{g}$ is continuous. So, $\mathbf{f} + \mathbf{g} \in C(0,1)$.
- (2. Commutativity): Clearly, $\mathbf{f} + \mathbf{g} = \mathbf{g} + \mathbf{f}$.
- (3. Associativity I): Clearly $(\mathbf{f} + \mathbf{g}) + \mathbf{h} = \mathbf{f} + (\mathbf{g} + \mathbf{h})$
- (4. The Zero): Let $\mathbf{f}_0$ denote the constant-zero function. So, $\mathbf{f}_0(x) = 0 \ \forall \ x \in \mathbb{R}$ So, $(\mathbf{f} + \mathbf{f}_0)(x) = \mathbf{f}(x) + 0 = \mathbf{f}(x)$. Therefore $\mathbf{f}_0$ satisfies the condition (4).

Continued

- (5. Additive inverse): Let $-\mathbf{f}$ denote the function $(-\mathbf{f})(x) = -\mathbf{f}(x)$. Then $(\mathbf{f} + (-\mathbf{f})) = 0 = \mathbf{f}_0$.
- (6. Closure under scalar multiplication): Clearly, $c\mathbf{f}$ is continuous and so $c\mathbf{f} \in C(0,1)$.
- (7. Distributivity I): Clearly, $c(\mathbf{f} + \mathbf{g}) = c\mathbf{f} + c\mathbf{g}$.
- (8. Distributivity II): Clearly, $(c + d)\mathbf{f} = c\mathbf{f} + d\mathbf{f}$.
- (9. Associativity II): Clearly, $c(d\mathbf{f}) = (cd)\mathbf{f}$
- (10. Multiplicative Identity):Clearly $1\mathbf{f} = \mathbf{f}$.

All 10 properties are verified. So, $C(0,1)$ is a vector space. ∎

A List of Important Vector Spaces

Here is a list of important vector spaces:

- $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, more generally $\mathbb{R}^n$ are vector spaces. Geometrically, $\mathbb{R}^2$ corresponds to the plane and $\mathbb{R}^3$ corresponds to the three dimensional space.
- $C(-\infty, \infty), C(a, b), C[a, b], C(a, b], C[a, b)$ the set of real valued continuous functions.
- P the set of all polynomials, with real coefficients
- P_n the set of all polynomials, with real coefficients, of degree less or equal to n.
- $M_{m,n}$ the set of all matrices of size $m \times n$ with real entries.

Theorem 4.2.1: Uniqueness of zero and $\mathbf{v}$

Theorem 4.2.1 Suppose V is a vector space. Then,

- There is exactly one vector satisfying the property of zero (Condition 4.) We say the additive identity $\mathbf{0}$ is unique.
- Given a vector $\mathbf{u}$ there is exactly one vector $\mathbf{x} \in V$, that satisfies condition 5. We say $\mathbf{u}$ has a unique additive inverse, to be denoted by denoted by $-\mathbf{u}$.

Continued

Proof. Suppose φ also satisfy condition 4. So, for any vector $u \in V$ we have

$$\mathbf{u} + \mathbf{0} = \mathbf{u} \quad \text{and} \quad \mathbf{u} + \varphi = \mathbf{u}.$$

Apply these two equations to $\mathbf{0}, \varphi$. We have

$$\varphi = \varphi + \mathbf{0} = \mathbf{0} + \varphi = \varphi.$$

So, the first statement is established.

Continued

To prove the second statement, assume, $\mathbf{x}, \mathbf{y} \in V$ satisfy condition 5, for $\mathbf{u}$. So.

$$\mathbf{u} + \mathbf{x} = \mathbf{0} \qquad \textit{and} \qquad \mathbf{u} + \mathbf{y} = \mathbf{0} \qquad \text{To prove :} \quad \mathbf{x} = \mathbf{y}.$$

We have (*using commutativity, Condition 2*)

$$\mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x} + (\mathbf{u} + \mathbf{y}) = (\mathbf{x} + \mathbf{u}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y}.$$

Properties of Scalar Multiplication

Theorem 4.2.2

Let $\mathbf{v}$ be a vector in a vector space V and c be a scalar. Then,

- (1) $0\mathbf{v} = \mathbf{0}$
- (2) $c\mathbf{0} = \mathbf{0}$
- (3)
$$c\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad c = 0 \quad or \quad \mathbf{v} = \mathbf{0}$$
- (4) $(-1)\mathbf{v} = -\mathbf{v}$.

Proof.

Proof.

- (1) By distributive property, We have $0\mathbf{v} + 0\mathbf{v} = (0+0)v = 0\mathbf{v}$. By (property 5), there is an additive inverse $-(0\mathbf{v})$ of $0\mathbf{v}$. We add the same to both sides of the above equation

$$(0\mathbf{v} + 0\mathbf{v}) + (-(0\mathbf{v})) = 0\mathbf{v} + (-(0\mathbf{v})) \qquad OR$$

$$0\mathbf{v} + (0\mathbf{v} + (-(0\mathbf{v}))) = \mathbf{0} \qquad OR$$

$$0\mathbf{v} + \mathbf{0} = \mathbf{0} \qquad Or \qquad 0\mathbf{v} = \mathbf{0}$$

So, (1) is established.

Continued

Proof.

- (2) First, by distributivity

$$c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$$

Now add $-(c\mathbf{0})$ to both sides:

$$c\mathbf{0} + (-(c\mathbf{0})) = (c\mathbf{0} + c\mathbf{0}) + (-(c\mathbf{0})) \qquad OR \qquad \mathbf{0} = c\mathbf{0}.$$

So, (2) is established.

Continued

Proof.

- (3) Suppose $c\mathbf{v} = \mathbf{0}$. Suppose $c \neq 0$. Then we can multiply the equation by $\frac{1}{c}$. So,

$$\frac{1}{c}(c\mathbf{v}) = \frac{1}{c}\mathbf{0} = \mathbf{0} \qquad (by\ (2))$$

By axion (10), we have

$$\mathbf{v} = 1\mathbf{v} = \left(\frac{1}{c}c\right)\mathbf{v} = \frac{1}{c}(c\mathbf{v}) = \frac{1}{c}\mathbf{0} = \mathbf{0}$$

So, either $c = 0$ or $\mathbf{v} = \mathbf{0}$ and (3) is established.

Continued

Proof.

- (4) We have, by distributivity and axiom (10)

$$\mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1-1)\mathbf{v} = 0\mathbf{v} = \mathbf{0} \quad \textit{by } (1).$$

  So, $(-1)\mathbf{v}$ satisfies the axiom (5) of the definition. So $(-1)\mathbf{v} = -\mathbf{v}$. So, (4) is established. ■

Examples of Sets with Operations that are not Vector Spaces

Given a set W and two operations (like addition and scalar myltiplication), it may fail to be a Vector Space for failure of any one or more of the axioms of the definition.

- **Example 5** The set W of all odd integers, with usual addition and scalar multiplication, is not a vector space. Note W is not closed under addition.
- **Example 6** The set of all integers $\mathbb{Z}$, with usual addition and scalar multiplication is not a vector space. Reason: $\mathbb{Z}$ is not closed under scalar multiplication $.5(1) \notin \mathbb{Z}$ not a Vector Space.

- **Example 7** Set S of polynomial of degree (exactly) 1 with usual addition and scalar multiplication is not a vector space. Reason:$\mathbb{Z}$ is not closed under Addition: $f(x) = 3x + 1, g(x) = -3x$ are in S. But $f + g = 1$ is not in S.
- **Example 8** Let $\mathbb{R}^2_1$ be a the set of all ordered pairs of (x, y) in the first quadrant. So $\mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 1\}$. Under usual addition ans scalar multiplication $\mathbb{R}^2_1$ is not a vector space. Reason: $(1, 1)$ does not have a additive inverse in $\mathbb{R}^2_1$.

Examples 4.2.1

Exercise 4.2.1 Describe the zero vector (additive identity) of $\mathbb{M}_{2,4}$.

Solution.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Examples 4.2.2

Examples 4.2.2 Let P_2 be the set of all polynomials of degree (exactly) 2. Is X a vector space? If not, why?
Solution. P_2 is not a vector space. (Here, by "degree 2" means, exactly of degree 2.)

$$\text{Let} \quad \mathbf{f}(x) = x^2 + x + 3, \quad \mathbf{g}(x) = -x^2 + 7x + 4$$

$$\textit{Then} \quad (\mathbf{f}+\mathbf{g})(x) = 8x + 7 \qquad \textit{has degree} \quad 1$$

$$\textit{So}, \quad \mathbf{f}, \mathbf{g} \in X, \quad \textit{but} \quad \mathbf{f}+\mathbf{g} \notin X.$$

So, X is not closed under addition.
Therefore X is not a vector space.

Example 4.2.3

Example 4.2.3

$$\textit{Let} \quad S = \left\{ \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Is it a vector space?

Solution

Yes, it is a vector space, because all the 10 conditions, of the definitions is satisfied:
(1) S is closed under addition, (2) addition commutes, (3) additions is associative, (4) S has the zero, (5) each matrix $\mathbf{v} \in S$, its $-\mathbf{v} \in S$ (6) S is claosed under scalar multiplication, (7) Distributivity I works, (8) Distributivity II works (9) Associativity II works, (10) $1\mathbf{v} = \mathbf{v}$. ■

Remark. A theorem will be proved in the next section, which states that we only need to check 2 contions: that S is closed under addition and scalar multiplication (condition 1 and 6).