Vector Spaces
§4.3 Subspaces of Vector Spaces

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Goals

- Define and discuss Subspaces of Vector Spaces
- Give examples

$$Q: What\ is\ a\ proof?$$

$$A: One-half\ \ percent\ of\ \ alcohol.$$

Subspaces:Definition

Numerous examples of Vector Spaces are "subspaces" of larger vector spaces.

Definition. Suppose V is a vector space. A non-empty subset W of V is called a subspace of V, if W is a vector space under the addition and scalar multiplication in V.

Example 4.3.1

Let

$$W_1 = \{(0, x_2, x_3, x_4) : x_1, x_2, x_3 \in \mathbb{R}\},$$

$$W_2 = \{(x_1, 0, x_3, x_4) : x_1, x_2, x_3 \in \mathbb{R}\},$$

$$W_3 = \{(x_1, x_2, 0, x_4) : x_1, x_2, x_3 \in \mathbb{R}\},$$

$$W_4 = \{(x_1, x_2, x_3, 0) : x_1, x_2, x_3 \in \mathbb{R}\},$$

Then, W_1, W_2, W_3, W_4 are subspaces of the $4-$space $\mathbb{R}^4$.
Proof. To prove W_1 is a subspace of $\mathbb{R}^4$, we need to check that all the 10 conditions of the definition of Vector Space is satisfied by W_1, which is routine checking.
However, the following theorem makes such proofs shorter. ■

Test for a subspace

Theorem 4.3.1 Suppose V is a vector space and W is a subset of V. Then, W is a subspace if and only if the following three conditions are satisfied:

- (1) W is non-empty (*notationally*, $W \neq \phi$).
- (2) If $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$.
  (*We say, W is closed under addition.*)
- (3) If $\mathbf{u} \in W$ and c is a scalar, then $c\mathbf{u} \in W$.
  (*We say, W is closed under scalar multiplication.*)

Proof. If W is a subspace, then it is a vector space by its won right. Hence, these three conditions holds, by definition of the same.

Conversely, assume that these three conditions hold. We need to check all 10 conditions are satisfied by W.

- Condition (1 and 6) are satisfied by hypothesis.
- W inherits condition (2, 3, 7, 8, 9) from the the "parent" vector space V.
- (Condition 4 : the Zero): Since $W \neq \phi$, there is a $\mathbf{w} \in W$. So, by (3), $\mathbf{0} = 0\mathbf{w} \in W$.
- (Condition 5): Negative on an element: Suppose $\mathbf{u} \in W$. The by (3), $-\mathbf{u} = (-1)\mathbf{u} \in W$.

Hence W is a vector space. ■

Trivial Subspaces

Let V be a vector space. Then,

1. V is a subspace of V.
2. Also, $\{\mathbf{0}\}$ is a subspace of V.
3. V and $\{\mathbf{0}\}$ may be called the trivial subspaces of V.

Examples 4.3.2

- **Example A** Recall $M_{n,n}$ denotes the vector space of all $n \times n$ matrices. A matrix A is defined to be a "symmetric matrix", if $A = A^T$. Let W be the set of all symmetric matrices of order n. Then W is a subspace of $M_{n,n}$.
- **Example B** $GL(n)$ be the set of all non-singular matrices of order n. Then, $GL(n)$ is a NOT subspace of $M_{n,n}$. This is because it is not closed under addition.

Examples 4.3.3: Subspaces of Functions

- $W_1 =$ set of all polynomial functions on $[0, 1]$.
- $W_2 =$ set of all differentiable (or smooth) functions on $[0, 1]$.
- $W_3 =$ set of all continuous functions on $[0, 1]$.
- $W_4 =$ set of all integrable functions on $[0, 1]$.
- $W_5 =$ set of all functions on $[0, 1]$.

$$Then, \quad W_1 \subseteq W_2 \subseteq W_3 \subseteq W_4 \subseteq W_5$$

All of them are vector spaces and each one is a subspace of the next one.

Intersection of Subspaces

- First, given two sets U, W the intersection $U \cap W$ is defined to be the set of all elements x that are in both U and W. Notationally,

$$U \cap W = \{x : x \in U, x \in W\}.$$

Theorem 4.3.2. Let U, W be two subspaces of a vector space V. Then, $U \cap W$ is a subspace of V.
Proof. We use theorem 4.3.1

- First, $\mathbf{0} \in U \cap W$. So, $U \cap W \neq \phi$.
- Suppose $\mathbf{x}, \mathbf{y} \in U \cap W$ and c is a scalar.

Continued

- Since U is closed under addition and scalar multiplication
$$\mathbf{x}+\mathbf{y} \in U, \qquad c\mathbf{x} \in U.$$
- For the same reason,
$$\mathbf{x}+\mathbf{y} \in W, \qquad c\mathbf{x} \in W.$$
- So,
$$\mathbf{x}+\mathbf{y} \in U \cap W, \qquad c\mathbf{x} \in U \cap W.$$
By theorem 4.3.1 $U \cap W$ is a subspace of U. The proof is complete. ■

Example 4.3.4: Subspaces of $\mathbb{R}^2$

- Let L be the set of all points on a line through the origin, in $\mathbb{R}^2$. Then, L is a subspace of $\mathbb{R}^2$. Recall, equation of a line through the origin is $ax + by = 0$.
- In particular, set of all points on $2x + 3y = 0$ is a subspace of $\mathbb{R}^2$.
- In particular, set of all points on $7x + 13y = 0$ is a subspace of $\mathbb{R}^2$.
- Set of all point on a line that does NOT pass through the origin, is NOT a subspace on $\mathbb{R}^2$. For example, $x + y = 1$ is such a line. This is because zero is not on this line.

Example 4.3.5: Subspaces of $\mathbb{R}^3$

Planes and lines through the origin are subspaces of $\mathbb{R}^3$.

- Let P be the set of all points on a plane through the origin, in 3−space $\mathbb{R}^3$. Then, P is a subspace of $\mathbb{R}^3$. Recall, equation of a plane through the origin is given by a homogeneous equation $ax + by + cz = 0$. For example, set of all points on $2x + 3y - 7z = 0$ is a subspace of $\mathbb{R}^3$.
- A plane P in $\mathbb{R}^3$ that does not pass through origin is NOT a subspace of $\mathbb{R}^3$. For example $2x + 3y - 7z = -1$ is NOT a subspace of $\mathbb{R}^3$. This is because zero is not on this plane.

- Let L be the set of all points on a line through the origin, in $\mathbb{R}^3$. Then, L is a subspace of $\mathbb{R}^3$. Recall, equation of a line through the origin is given by ahomogeneous system of "independent" linear equations

$$\begin{cases} a_1x & +b_1y & +c_1z & = 0 \\ a_2x & +b_2y & +c_2z & = 0 \end{cases}$$

- In particular, set of all points on the line:

$$\begin{cases} x & +2y & +3z & = 0 \\ x & +y & +2z & = 0 \end{cases}$$

is a subspace of $\mathbb{R}^3$:

- A line in $\mathbb{R}^3$ that does NOT pass through the origin, is NOT a subspace of $\mathbb{R}^3$. For example, the set L of all points on the line:

$$\begin{cases} x & +2y & +3z & = 1 \\ x & +y & +2z & = 0 \end{cases}$$

does NOT form a subspace of $\mathbb{R}^3$

Example 4.3.6: Subspaces of $\mathbb{R}^n$

- Consider the homogeneous Lineear system

$$A\mathbf{x} = \mathbf{0} \quad A \text{ is a } m \times n \text{ matrix, and } \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

Let H be the set of all solutions of this system. Then, H is a subspace of $\mathbb{R}^n$.

- In the geometric language, such a subspace H is a hyper-plane in $\mathbb{R}^n$, passing through the origin.

Example 4.3.7

$$Let \quad W = \left\{ \begin{pmatrix} a & b & 0 \\ a+b & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

Is W a subspace of $V = \mathbb{M}_{2,3}$?
Solution. The answer is: Yes, it is a subspace. We need to check there properties of the theorem above.

- First, $\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in W$. So, W is non-empty.

Continued

- W is closed under addition: Let

$$\mathbf{x} = \begin{pmatrix} a & b & 0 \\ a+b & 0 & c \end{pmatrix}, \mathbf{y} = \begin{pmatrix} x & y & 0 \\ x+y & 0 & z \end{pmatrix} \in W.$$

So, $$\mathbf{x}+\mathbf{y} = \begin{pmatrix} a+x & b+y & 0 \\ (a+x)+(b+y) & 0 & c+z \end{pmatrix} \in W$$

because it has the same form as elements of W.

Continued

- W is closed under : Let $\mathbf{x} \in W$ be as above and s be a scalar.

$$Then, \qquad s\mathbf{x} = \begin{pmatrix} sa & sb & 0 \\ sa + sb & 0 & sc \end{pmatrix}$$

because it has the same form as elements of W. So, all three conditions of the theorem is satisfied. So, W is a subspace of V.

Non-Example 4.3.8

Let W be the set of all vectors in $\mathbb{R}^2$ whose components are rational numbers. Is W a subspace of $\mathbb{R}^2$
Solution. Then answers is "NO": W is not closed under scalar multiplication. $\mathbf{x} = (1, 0) \in W$. But $\pi\mathbf{x} = (\pi, 0) \notin W$. So, W is not a vector space over $\mathbb{R}$.

Remark. W will be a vector space over the rationals $\mathbb{Q}$. However, in this class we talk only about vector spaces of $\mathbb{R}$.