Chapter 5: Eigenvalues and Eigenvectors
§5.1 Eigenvalues and Eigenvectors

Satya Mandal, KU

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Goals

Suppose A is square matrix of order n.

- Eigenvalues of A will be defined.
- Eigenvectors of A, corresponding to each eigenvalue, will be defined.
- Eigenspaces of A, corresponding to each eigenvalue, will be defined.

Definitions

Definition Suppose A is square matrix of order n. A scalar λ (real of complex) is said to an eigenvalue of A, if

$$\text{there is an } \mathbf{x} \in \mathbb{C}^n, \ \mathbf{x} \neq \mathbf{0} \ \text{ such that } \ (\lambda I - A)\mathbf{x} = \mathbf{0}.$$

Such an $\mathbf{x} \in \mathbb{C}^n$ is called an eigenvector of A, corresponding to λ. (*Note, the zero vector* $\mathbf{0}$ *is not considered an eigenvector.*)

- So, an eigenvalue of λ, can be a real or a complex.
- Complex eigenvalues are often avoided in this course, while it is useful in the DE course.
- **Remark.** "eigen" is a German word, meaning "characteristic".

Theorem 5.1.1

Theorem 5.1.1: Let A be a square matrix of order n. Let λ be a number (real or complex) Then, λ is an eigenvalue of A if and only if $|\lambda I - A| = 0$.

Proof. First, consider the case, when $\lambda \in \mathbb{R}$ is real.

- Assume $|\lambda I - A| = 0$. Then, it follows that the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial (i.e. nonzero) solution $\mathbf{x} \in \mathbb{R}^n$. So, λ is an eigenvalue.
- Conversely, assume that λ is an eigenvalue of A. By definition, $(\lambda I - A)\mathbf{x} = \mathbf{0}$, for some $\mathbf{x} \in \mathbb{C}^n$. with $\mathbf{x} \neq \mathbf{0}$. We can write $\mathbf{x} = \mathbf{u} + i\mathbf{v}$. Since λ is real, it follows $(\lambda I - A)\mathbf{u} = (\lambda I - A)\mathbf{v} = \mathbf{0}$. Since, either $\mathbf{u} \neq 0$ or $\mathbf{v} \neq 0$, the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nonzero real solution. It follows, $|\lambda I - A| = 0$.

Continued

- This completes the proof of the theorem, in this case.
- When λ is a complex number, the proof follows from exactly the same theory.(*We have been shy to deal with complex numbers.*)

Eigenspaces: Definition

Definition: Suppose A is a square matrix, of order n. Let λ be an eigenvalue of A.
The Eigenspace $E(\lambda)$, of A corresponding to λ, is defined to be set of all solutions

$$E(\lambda) = \begin{cases} \{\mathbf{x} \in \mathbb{R}^n : (\lambda I - A)\mathbf{x} = \mathbf{0}\} & \text{if } \lambda \text{ is real} \\ \{\mathbf{x} \in \mathbb{C}^n : (\lambda I - A)\mathbf{x} = \mathbf{0}\} & \text{if } \lambda \text{ is complex} \end{cases}$$

Note, $E(\lambda)$ consisits of all the eigen vectors of λ and $\mathbf{0}$.

Eigenspaces: Theorem 5.1.2

Suppose A is a square matrix, of order n. Let λ be an eigenvalue of A.
If λ is real, then $E(\lambda)$ is a subspace of $\mathbb{R}^n$.
If λ is complex, then $E(\lambda)$ is a subspace of $\mathbb{C}^n$.
Proof. Assume λ is real. $E(\lambda)$ is the null space of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$. Then, by Theorem 4.6.4 4 $E(\lambda)$ is a subspace.
When λ is complex, we avoided the theory. However, proof is similar. The proof is complete. ■

The Characteristic Equation

Definition. Let A be a square matrix of order n.

- **Definition:** Then the equation

$$|\lambda I - A| = 0$$

is called the characteristic equation of A.

- **Definition:** Expanding the determinant $|\lambda I - A|$, it follows

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$

which is a polynomial in λ, of degree n. This polynomial is called the characteristic polynomial of A.

Computing

- To compute the eigenvalues of A, solve the characteristic equation
$$|\lambda I - A| = 0.$$
So, we expect both real and complex eigen values.
- Given an eigenvalue λ_i to compute the eigenspace $E(\lambda_i)$, solve the linear system
$$(\lambda_i I - A)\mathbf{x} = \mathbf{0}.$$
Since λ_i is an eigenvalue, this is a singular system. Solve it by row reduction.

Theorem 5.1.3

Theorem 5.1.3 If A is a diagonal matrix, then its eigenvalues are the diagonal entries.

Proof. Let

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \quad \textit{be a diagonal matrix}.$$

Then, characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & 0 & \cdots & 0 \\ 0 & \lambda - a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda - a_{nn} \end{vmatrix} = 0$$

Continued

Which is

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = 0$$

So, the eigenvalues are

$$\lambda = a_{11}, a_{22}, \cdots, a_{nn}$$

■

Theorem. If A is a triangular matrix, then its eigenvalues are the diagonal entries.
Proof. Similar to the above.

Example 5.1.1

Let $A = \begin{pmatrix} 3 & 5 & -3 \\ 6 & 2 & -3 \\ 6 & 5 & -6 \end{pmatrix}$. Verify that $\lambda_1 = 5$ is an eigenvalue of A and $\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a corresponding eigenvector.

Continued

Solution: Because of Theorem 5.1.1, it would suffice to check $A\mathbf{x_1} = 5\mathbf{x_1}$, We have

$$A\mathbf{x_1} = \begin{pmatrix} 3 & 5 & -3 \\ 6 & 2 & -3 \\ 6 & 5 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 5\mathbf{x_1}.$$

So, assertion is verified. ■

Continued

Verify that $\lambda_2 = -3$ is a eigenvalue of A and $\mathbf{x_2} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ is a corresponding eigenvector.

Solution: Because of Theorem 5.1.1, it would suffice to check $A\mathbf{x_2} = -3\mathbf{x_2}$, We have

$$A\mathbf{x_2} = \begin{pmatrix} 3 & 5 & -3 \\ 6 & 2 & -3 \\ 6 & 5 & -6 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ -3 \end{pmatrix}$$

$$= -3 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = -3\mathbf{x_2}.$$

So, assertion is verified.

Continued

Verify that $\lambda_3 = -3$ is a eigenvalue of A and $\mathbf{x_3} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ is a corresponding eigenvector.

Solution: Because of Theorem 5.1.1, it would suffice to check $A\mathbf{x_3} = -3\mathbf{x_3}$, We have $A\mathbf{x_3} =$

$$\begin{pmatrix} 3 & 5 & -3 \\ 6 & 2 & -3 \\ 6 & 5 & -6 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ 9 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = -3\mathbf{x_3}.$$

So, assertion is verified. ■

Remark: Continued

- Note, the matrix A has, at least, two distinct eigenvalues $\lambda = 5$, $\lambda = -3$.
- Further, corresponding to $\lambda = -3$, we have exhibited two eigenvectors $\mathbf{x}_2$ and $\mathbf{x}_3$, which are linearly independent (check).

Example 5.1.2

Let

$$A = \begin{pmatrix} -4 & -5 & 5 \\ -2 & -1 & -1 \\ -16 & -17 & 13 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Determine whether **x** is an eigenvector of A.

Solution: We have

$$A\mathbf{x} = \begin{pmatrix} -4 & -5 & 5 \\ -2 & -1 & -1 \\ -16 & -17 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

for all λ. So, **x** is not an eigenvector of A.

Example 5.1.2b

Determine whether $\mathbf{x} = \begin{pmatrix} -5 \\ 7 \\ 3 \end{pmatrix}$ is an eigenvector of A.

Solution: We have

$$A\mathbf{x} = \begin{pmatrix} -4 & -5 & 5 \\ -2 & -1 & -1 \\ -16 & -17 & 13 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \\ 3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} = 0\mathbf{x}.$$

So, $\mathbf{x}$ is an eigenvector and corresponding eigenvalue is $\lambda = 0$.

■

Example 5.1.2c

Determine whether $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of A.

Solution: No, $\mathbf{0}$ is, by definition, never an eigenvector.

Example 5.1.2d

Determine whether

$$\mathbf{x} = (2\sqrt{6} - 3, -2\sqrt{6} + 6, 3)^T = \begin{pmatrix} 2\sqrt{6} - 3 \\ -4\sqrt{6} + 9 \\ -2\sqrt{6} + 9 \end{pmatrix}$$ is an

eigenvector of A.

Solution: We have

$$A\mathbf{x} = \begin{pmatrix} -4 & -5 & 5 \\ -2 & -1 & -1 \\ -16 & -17 & 13 \end{pmatrix} \begin{pmatrix} 2\sqrt{6} - 3 \\ -4\sqrt{6} + 9 \\ -2\sqrt{6} + 9 \end{pmatrix} = \begin{pmatrix} 2\sqrt{6} + 12 \\ 2\sqrt{6} - 12 \\ 10\sqrt{6} + 12 \end{pmatrix}$$

$$= (2\sqrt{6}+4)\begin{pmatrix} 2\sqrt{6}-3 \\ -4\sqrt{6}+9 \\ -2\sqrt{6}+9 \end{pmatrix} = A\mathbf{x}$$

So, $\mathbf{x}$ is an eigenvector of A, for the eigenvalue $\lambda = 2\sqrt{6}+4$.

■

Example 5.1.3

Let

$$A = \begin{pmatrix} -5 & 0 & 0 \\ -1 & 7 & 0 \\ -1 & 1 & 3 \end{pmatrix}.$$

(a) Find the characteristic equation of A, (b) Find all the eigenvalues of A, (c) Corresponding to each eigenvalue, compute the eigenspace.

Example 5.1.3: Solution

Solution: The characteristic polynomial is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+5 & 0 & 0 \\ 1 & \lambda-7 & 0 \\ 1 & -1 & \lambda-3 \end{vmatrix} = (\lambda+5)(\lambda-7)(\lambda-3).$$

So, the characteristic equation is

$$(\lambda+5)(\lambda-7)(\lambda-3) = 0.$$

Therefore, the eigenvalues are $\lambda = -5, 7, 3.$.

Continued

To find an eigenvector corresponding to $\lambda = -5$, solve $(-5I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -12 & 0 \\ 1 & -1 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving, we get

$$x = t, \qquad y = \frac{1}{12}t \qquad z = \frac{1}{8}x - \frac{1}{8}y = \frac{11}{96}t$$

Continued

So, the eigenspace of $\lambda = -5$ is

$$\left\{ \begin{pmatrix} 1 \\ \frac{1}{12} \\ \frac{11}{96} \end{pmatrix} t : t \in \mathbb{R} \right\}.$$

In particular, with $t = 1$, $\begin{pmatrix} 1 \\ \frac{1}{12} \\ \frac{11}{96} \end{pmatrix}$ is an eigenvector of A , corresponding to the eigenvalue $\lambda = -5$.

Continued

To find an eigenvector corresponding to $\lambda = 7$, wehave to solve $(7I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} 12 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving, we get

$$x = 0 \qquad y = t \qquad z = \frac{1}{4}(y - x) = \frac{1}{4}t.$$

Continued

So, that eigenspace of $\lambda = 7$ is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix} t : t \in \mathbb{R} \right\}.$$

In particular, with $t = 1$, $\begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix}$ is an eigenvector of A, for eigenvalue $\lambda = 7$.

Continued

To find an eigenvector corresponding to $\lambda = 3$, wehave to solve $(3I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} 8 & 0 & 0 \\ 1 & -4 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving, we get

$$x = 0 \qquad y = \frac{1}{4}x = 0 \qquad z = t.$$

So, that eigenspace of $\lambda = 3$ is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t : t \in \mathbb{R} \right\}.$$

In particular, with $t = 1$, an eigenvector for eigenvalue $\lambda = 3$

Example 5.1.4

Let

$$A = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & -1 & 4 \end{pmatrix}.$$

Find the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 4$.

Solution: The eigenspace $E(3)$ is the solution space of the system $(4I - A)\mathbf{x} = \mathbf{x}$, or

$$\begin{pmatrix} 4-4 & 0 & 0 \\ 1 & 4-4 & 0 \\ 1 & 1 & 4-4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Continued

or

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The coefficient matrix

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad has\ rank\ 2.$$

$$Since \quad rank(C) + nullity(C) = 3, \qquad nullity(C) = 1.$$

Therefore, $\dim E(3) = nullity(C) = 1$.