Eigenvalues and Eigenvectors §5.2 Diagonalization

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Goals

Suppose A is square matrix of order n.

- Provide necessary and sufficient condition when there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Definitions

- Two square matrices A, B are said to be similar, if there is an invertible matrix P, such that $A = P^{-1}BP$.
- A square matrix A said to be diagonalizable, if there is an invertible matrix P, such that $P^{-1}AP$ is a diagonal matrix. That means, if A is similar to a diagonal matrix, we say that A is diagonalizable.

Theorem 5.2.1

Suppose A, B are two similar matrices. Then, A and B have same eigenvalues.

Proof. Write $A = P^{-1}BP$. Then

$$|\lambda I - A| = |\lambda I - P^{-1}BP| = |\lambda(P^{-1}P) - P^{-1}BP| = |P^{-1}(\lambda I - B)P|$$

$$= |P^{-1}||\lambda I - B||P| = |P|^{-1}|\lambda I - B||P| = |\lambda I - B|$$

So, A and B has same characteristic polynomials. So, they have same eigenvalues. The proof is complete. ■

Theorem 5.2.2: Diagonalizability

We ask, when a square matrix is diagonalizable?

Theorem 5.2.2 A square matrix A, of order n, is diagonalizable if and only if A has n linearly independent eigenvectors.

Proof. There are two statements to prove. First, suppose A is diagonalizable.

$$\text{Then } P^{-1}AP = D, \quad \text{and hence} \quad AP = PD$$

where P is an invertible matrix and D is a diagonal matrix.

$$\text{Write, } D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \; P = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{pmatrix}$$

Continued

- Since $AP = PA$, we have

$$A \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Or

$$\begin{pmatrix} A\mathbf{p_1} & A\mathbf{p_2} & \cdots & A\mathbf{p_n} \end{pmatrix} = \begin{pmatrix} \lambda_1\mathbf{p_1} & \lambda_2\mathbf{p_2} & \cdots & \lambda_n\mathbf{p_n} \end{pmatrix}$$

Continued

- So,
  $$A\mathbf{p_i} = \lambda_i \mathbf{p_i} \qquad for \quad i = 1, 2, \cdots, n$$
  Since P is invertible, $\mathbf{p_i} \neq \mathbf{0}$ and hence $\mathbf{p_i}$ is an eigenvector of A, for λ.
- Also, $rank(P) = n$. So, its columns $\{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\}$ are linearly independent.
- So, it is established that if A is diagonalizable, then A has n linearly independent eigenvectors.

Continued

- Now, we prove the converse. So, we assume A bas has n linearly independent eigenvectors:

$$\{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\}$$

- So,

$$A\mathbf{p_1} = \lambda_1 \mathbf{p_1}, A\mathbf{p_2} = \lambda_2 \mathbf{p_2}, \cdots, A\mathbf{p_n} = \lambda_n \mathbf{p_n} \quad \textit{for some} \quad \lambda_i.$$

Continued

- Write,

$$P = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{pmatrix} \text{ and } D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

- It follows from the equations $A\mathbf{p_i} = \lambda_i \mathbf{p_i}$ that

$$AP = PD. \quad So, \quad P^{-1}AP = D \quad is\ diagonal.$$

The proof is complete. ■

Steps for Diagonalizing

Suppose A is a square matrix of order n.

- If A does not have n linearly independent eigenvectors, then A is not diagonalizable.
- When possible, find n linearly independent eigenvectors $\mathbf{p_1}, \mathbf{p_2}, \cdots, \mathbf{p_n}$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- Then, write

$$P = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{pmatrix} \text{ and } D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

- We have $D = P^{-1}AP$ is a diagonal matrix.

Corollary 4.4.3

Suppose V is a vectors space and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be vectors in V. Then, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly dependent if and only if there is an integer $m \leq n$ such that (1) $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly dependent and (2) $\mathbf{x}_m \in span(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{m-1})$.

Proof. Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly dependent. By Theorem 4.4.2, one of these vectors is a linear combination of the rest. By relabeling, we can assume $\mathbf{x}_n$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}$. Let

$$m = \min\{k : \mathbf{x}_k \in span(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})\}$$

Continued

If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{m-1}$ are linearly dependent, then we could apply Theorem 4.4.2 again, which would lead to a contradiction, that m is minimum. So, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{m-1}$ are linearly independent. This establishes one way implication.
Conversely, suppose there is an $m \leq n$ such that (1) and (2) holds. Then,

$$\mathbf{x}_m = c_1\mathbf{x}_1 + \cdots + c_{m-1}\mathbf{x}_{m-1} \quad \text{for some} \quad c_1, \ldots, c_{m-1} \in \mathbb{R}$$

So,

$$c_1\mathbf{x}_1 + \cdots + c_{m-1}\mathbf{x}_{m-1} + (-1)\mathbf{x}_m = \mathbf{0}$$

which is a nontrivial linear combination. So, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \ldots, \mathbf{x}_n$ are linearly dependent. ■

Theorem 5.2.3: With Distinct Eigenvalues

Let A be a square matrix A, of order n. Suppose A has n distinct eigenvalues. Then

- the corresponding eigenvectors are linearly independent
- and A is diagonalizable.

Proof.

- The second statement follows from the first, by theorem 5.2.2. So, we prove the first statement only.
- Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of A.
- So, for $i = 1, 2, \ldots, n$ we have

$$A\mathbf{x_i} = \lambda_i \mathbf{x_i} \quad where \quad \mathbf{x_i} \neq 0 \quad are\ eigenvectors.$$

Continued

- We need to prove that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent. We prove by contra-positive argument.
  - So, assume they are linearly dependent.
  - By Corollary 4.4.3 there is an $m < n$ such that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ are mutually linearly independent and $\mathbf{x_{m+1}}$ is in can be written as a linear combination of $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}\}$. So,

$$\mathbf{x_{m+1}} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_m\mathbf{x_m} \qquad (1)$$

Here, at least one $c_i \neq 0$. Re-labeling $\mathbf{x_i}$, if needed, we can assume $c_1 \neq 0$.

Continued

- Multiply (1) by A on the left:

$$A\mathbf{x_{m+1}} = c_1 A\mathbf{x_1} + c_2 A\mathbf{x_2} + \cdots + c_m A\mathbf{x_m} \qquad (2)$$

Now, use $A\mathbf{x_i} = \lambda_i \mathbf{x_i},$:

$$\lambda_{m+1}\mathbf{x_{m+1}} = \lambda_1 c_1 \mathbf{x_1} + \lambda_2 c_2 \mathbf{x_2} + \cdots + \lambda_m c_m \mathbf{x_m} \qquad (3)$$

- Also, multiply (1) by λ_{m+1}, we have

$$\lambda_{m+1}\mathbf{x_{m+1}} = \lambda_{m+1} c_1 \mathbf{x_1} + \lambda_{m+1} c_2 \mathbf{x_2} + \cdots + \lambda_{m+1} c_m \mathbf{x_m} \quad (4)$$

Continued

- Subtract (3) from (4):

$$(\lambda_{m+1}-\lambda_1)c_1\mathbf{x_1}+(\lambda_{m+1}-\lambda_2)c_2\mathbf{x_2}+\cdots+(\lambda_{m+1}-\lambda_m)c_m\mathbf{x_m}=\mathbf{0}.$$

- Since these vectors are linearly independent, and hence

$$(\lambda_{m+1}-\lambda_i)c_i=0 \qquad \mathit{for} \quad i=1,2,\cdots,m.$$

- Since $c_1 \neq 0$ we get $\lambda_{m+1}-\lambda_1=0$ or $\lambda_{m+1}=\lambda_1$. This contradicts that λ_is are distinct. So, we conclude that $\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_n}$ are linearly independent. The proof is complete. ■

Example 5.2.2

$$Let \quad A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \quad and \quad P = \begin{pmatrix} 1 & 1 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Verify that A is diagonalizable, by computing $P^{-1}AP$.

Solution: We do it in a two steps.

1. Use TI to compute

$$P^{-1} = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -1 & .5 \\ 0 & 0 & .5 \end{pmatrix}. \ So, \quad P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

So, it is verified that $P^{-1}AP$ is a diagonal matrix.

Example 5.2.3

$$Let \qquad A = \begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix}.$$

Show that A is not diagonalizable.

Solution: Use Theorem 5.2.2 and show that A does not have 2 linearly independent eigenvectors. To do this, we have find and count the dimensions of all the eigenspaces $E(\lambda)$. We do it in a few steps.

- First, find all the eigenvalues. To do this, we solve

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 \\ 9 & \lambda + 3 \end{vmatrix} = \lambda^2 = 0.$$

  So, $\lambda = 0$ is the only eigenvalue of A.

Continued

- Now we compute the eigenspace $E(0)$ of the eigenvalue $\lambda = 0$. We have $E(0)$ is solution space of

$$(0I - A)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -3 & -1 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using TI (or by hand), a parametric solution of this system is given by $x = -.5t \quad y = t$.

$$So \quad E(0) = \{(t, -3t) : t \in \mathbb{R}\} = \mathbb{R}1, -3).$$

So, the (sum of) dimension(s) of the eigenspace(s)

$$= \dim E(0) = 1 < 2.$$

Therefore A is not diagonalizable.

Example 5.2.3

$$\text{Let} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}.$$

Show that A is not diagonalizable.

Solution: Use Theorem 5.2.2 and show that A does not have 3 linearly independent eigenvectors.

- To find the eigenvalues, we solve

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda + 3 & -1 \\ 0 & 0 & \lambda + 3 \end{vmatrix} = (\lambda - 1)(\lambda + 3)^2 = 0.$$

So, $\lambda = 1, -3$ are the only eigenvalues of A.

Continued

- We have $E(1)$ is solution space of

$$(I-A)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Or} \quad \begin{pmatrix} 0 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(*As an alternative approach, avoid solving this system.*) The (column) rank of the coefficient matrix is 2. So, $\dim(E(1)) = nullity = 3 - rank = 3 - 2 = 1$.

Continued

- Now we compute the dimension $\dim E(-3)$. $E(-3)$ is the solution space of

$$(-3I - A)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad or$$

$$\begin{pmatrix} -4 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of the coefficient matrix is 2 (use TI, if you need). So,

$$\dim(E(-3)) = nullity = 3 - rank = 3 - 2 = 1.$$

Continued

- So, the sum of dimensions of the eigenspaces

$$= \dim E(1) + \dim E(-3) = 2 < 3.$$

Therefore A is not diagonalizable.

Example 5.2.4

$$Let\quad A=\begin{pmatrix} 17 & 113 & -2 \\ 0 & \sqrt{2} & 1 \\ 0 & 0 & \pi \end{pmatrix} \quad \text{Find its eigenvalues}$$

and determine (use Theorem 5.2.3), if A is diagonalizable. If yes, write down a an invertible matrix P so that $P^{-1}AP$ is a diagonal matrix.

Solution: To find eigenvalues solve

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-17 & -113 & 2 \\ 0 & \lambda-\sqrt{2} & -1 \\ 0 & 0 & \lambda-\pi \end{vmatrix}$$

$$= (\lambda-17)(\lambda-\sqrt{2})(\lambda-\pi) = 0.$$

Continued

So, A has three distinct eigenvalues $\lambda = 17, \sqrt{2}, \pi$. Since A is a 3×3 matrix, by Theorem 5.2.3 , A is diagonalizable.
We will proceed to compute the matrix P, by computing bases of $E(17)$, $E(\sqrt{(2)})$ and $E(\pi)$. ∎

Continued

To compute $E(17)$, we solve: $(17I_3 - A)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 0 & -113 & 2 \\ 0 & 17-\sqrt{2} & -1 \\ 0 & 0 & 17-\pi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $z = y = 0$ and $x = t$, for any $t \in \mathbb{R}$. So,

$$E(17) = \left\{ \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Continued

$$\text{with } t = 1 \text{ a basis of } E(17) \text{ is } \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Continued

To compute $E(\sqrt{2})$, we solve: $(\sqrt{2}I_3 - A)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} \sqrt{2}-17 & -113 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & \sqrt{2}-\pi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $z = 0$ and $x = t$ and $y = \frac{\sqrt{2}-17}{113}t$ for any $t \in \mathbb{R}$. So,

$$E(\sqrt{2}) = \left\{ \begin{pmatrix} t \\ \frac{\sqrt{2}-17}{113}t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Continued

$$\text{with } t = 113 \text{ a basis of } E(\sqrt{2}) \text{ is } \left\{ \begin{pmatrix} 113 \\ \sqrt{2} - 17 \\ 0 \end{pmatrix} \right\}$$

Continued

To compute $E(\pi)$, we solve: $(\pi I_3 - A)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} \pi - 17 & -113 & 2 \\ 0 & \pi - \sqrt{2} & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} z = t \\ y = \frac{1}{\pi - \sqrt{2}} z = \frac{1}{\pi - \sqrt{2}} t \\ x = \frac{113}{\pi - 17} y - \frac{2}{\pi - 17} z = \frac{113 + 2\sqrt{2} - 2\pi}{(\pi - 17)(\pi - \sqrt{2})} t \end{cases}$$

$$E(\pi) = \left\{ \begin{pmatrix} \frac{113 + 2\sqrt{2} - 2\pi}{(\pi - 17)(\pi - \sqrt{2})} t \\ \frac{1}{\pi - \sqrt{2}} t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Continued

With $t = (\pi - 17)(\pi - \sqrt{2})$ a basis of $E(\pi)$ is

$$\left\{ \begin{pmatrix} 113 + 2\sqrt{2} - 2\pi \\ \pi - 17 \\ (\pi - 17)(\pi - \sqrt{2}) \end{pmatrix} \right\}$$

Continued

We form the matrix of the eigenvectors.

$$P = \begin{pmatrix} 1 & 113 & 113 + 2\sqrt{2} - 2\pi \\ 0 & \sqrt{2} - 17 & \pi - 17 \\ 0 & 0 & (\pi - 17)(\pi - \sqrt{2}) \end{pmatrix}.$$

We check

$$P^{-1}AP = \begin{pmatrix} 17 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \pi \end{pmatrix} \quad \text{Or} \quad AP = P \begin{pmatrix} 17 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \pi \end{pmatrix}$$

Continued

We have

$$AP = \begin{pmatrix} 17 & 113 & -2 \\ 0 & \sqrt{2} & 1 \\ 0 & 0 & \pi \end{pmatrix} \begin{pmatrix} 1 & 113 & 113+2\sqrt{2}-2\pi \\ 0 & \sqrt{2}-17 & \pi-17 \\ 0 & 0 & (\pi-17)(\pi-\sqrt{2}) \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 113\sqrt{2} & \pi(113+2\sqrt{2}-2\pi) \\ 0 & \sqrt{2}(\sqrt{2}-17) & \pi(\pi-17) \\ 0 & 0 & \pi(\pi-17)(\pi-\sqrt{2}) \end{pmatrix}$$

$$= P \begin{pmatrix} 17 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \pi \end{pmatrix}$$