Inner Product Spaces
§6.1 Length and Dot Product in $\mathbb{R}^n$

Satya Mandal, KU

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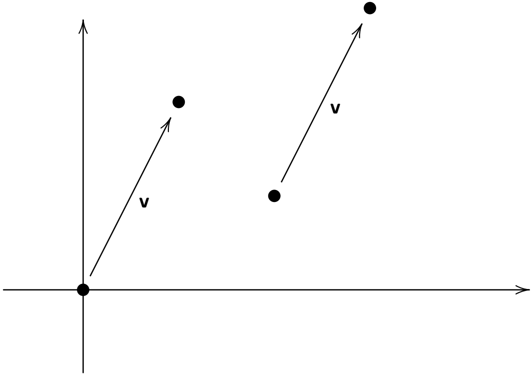
Goals

We imitate the concept of length and angle between two vectors in $\mathbb{R}^2, \mathbb{R}^3$ to define the same in the $n-$space $\mathbb{R}^n$. Main topics are:

- Length of vectors in $\mathbb{R}^n$.
- Dot product of vectors in $\mathbb{R}^n$ (It comes from angles between two vectors).
- Cauchy Swartz Inequality in $\mathbb{R}^n$.
- Triangular Inequality in $\mathbb{R}^n$, like that of triangles.

Length and Angle in plane $\mathbb{R}^2$



- We discussed, two parallel arrows, with equal length, represented the Same Vector **v**.
- In particular, there is one arrow, representing **v**, starting at the origin.

Continued

- Such arrows, starting at the origin, are identified with points (x, y) in $\mathbb{R}^2$. So, we write $\mathbf{v} = (v_1, v_2)$.
- The length of the vector $\mathbf{v} = (v_1, v_2)$ is given by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}.$$

- Also, the angle θ between two such vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ is given by

$$\cos\theta = \frac{v_1 u_1 + v_2 u_2}{\|\mathbf{v}\| \, \|\mathbf{u}\|}$$

- Subsequently, we imitate these two formulas.

Length on $\mathbb{R}^n$

Definition. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a vector in $\mathbb{R}^n$.

- The length or magnitude or norm of $\mathbf{v}$ is defined as

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

- So, $\|\mathbf{v}\| = 0 \iff \mathbf{v} = 0.$
- We say $\mathbf{v}$ is a unit vector if $\|\mathbf{v}\| = 1.$

Theorem 6.1.1: Length in $\mathbb{R}^n$

Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a vector in $\mathbb{R}^n$ and $c \in \mathbb{R}$ be a scalar. Then $\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$.

Proof.

- We have $c\mathbf{v} = (cv_1, cv_2, \ldots, cv_n)$.
- Therefore, $\|c\mathbf{v}\| =$

$$\sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$

$$= \sqrt{c^2 \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)} = |c| \, \|\mathbf{v}\| .$$

The proof is complete. ■

Theorem 6.1.2: Length in $\mathbb{R}^n$

Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a non-zero vector in $\mathbb{R}^n$. Then,

$$\mathbf{u} = \frac{\mathbf{v}}{\parallel \mathbf{v} \parallel}$$

has length 1. We say, **u** is the **unit vector in the direction of v.**

Proof. (*First, note that the statement of the theorem would not make sense. unless* **v** *is nonzero.*) Now,

$$\|\mathbf{u}\| = \left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = 1.$$

The proof is complete. ■

Comments

- **Example.** The standard basis vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_2 = (0,0,1) \in \mathbb{R}^3$ are unit vectors in $\mathbb{R}^3$.
- **Example.** Similarly, recall the standard basis of $\mathbb{R}^n$

$$\begin{cases} \mathbf{e}_1 = (1,0,0,\ldots,0) \\ \mathbf{e}_2 = (0,1,0,\ldots,0) \\ \mathbf{e}_3 = (0,0,1,\ldots,0) \\ \cdots \\ \mathbf{e}_n = (0,0,0,\ldots,1) \end{cases} \tag{1}$$

Here, each $\mathbf{e}_i$ is a unit vectors in $\mathbb{R}^n$.

Continued: Direction

- For a nonzero vector $\mathbf{v}$ and scalar $c > 0$ $c\mathbf{v}$ points to the same direction as $\mathbf{v}$ and $-c\mathbf{v}$ point to direction opposite to $\mathbf{v}$.

Distance

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbb{R}^n$. Then, the distance between $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

it is easy to see:

1. $d(\mathbf{u}, \mathbf{v}) \geq 0$.
2. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.
3. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.

Example 6.1.1

Let $\mathbf{u} = (1, 2, 2), \quad \mathbf{v} = (-3, 1, -2)$.

1. Compute $\| \mathbf{u} \|, \| \mathbf{v} \|, \| \mathbf{u} + \mathbf{v} \|$. **Solution:**

$$\| \mathbf{u} \| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3.$$

$$\| \mathbf{v} \| = \sqrt{(-3)^2 + 1^2 + (-2)^2} = \sqrt{14}.$$

$$\| \mathbf{u} + \mathbf{v} \| = \sqrt{(1-3)^{+}(2+1)^2 + (2-2)^2} = \sqrt{13}.$$

2. Compute distance $d(\mathbf{u}, \mathbf{v})$. **Solution:**

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1+3)^2 + (2-1)^2 + (2+2)^2} = \sqrt{33}$$

Example 6.1.2

Let $\mathbf{u} = (-1, \sqrt{10}, 3, 4)$.

1. Compute the unit vector in the direction of $\mathbf{u}$. **Solution:** First, $\| \mathbf{u} \| = \sqrt{(-1)^2 + (\sqrt{10})^2 + 3^2 + 4^2} = 6$. The unit vector in the direction of $\mathbf{u}$ is

$$\mathbf{e} = \frac{\mathbf{u}}{\| \mathbf{u} \|} = \frac{(-1, \sqrt{10}, 3, 4)}{6} = \left(-\frac{1}{6}, \frac{\sqrt{10}}{6}, \frac{3}{6}, \frac{4}{6} \right).$$

2. Compute the unit vector in the direction opposite of $\mathbf{u}$. **Solution:** Answer is $-\mathbf{e} = \left(\frac{1}{6}, \frac{-\sqrt{10}}{6}, \frac{-3}{6}, \frac{-4}{6} \right)$.

Example 6.1.3

Let $\mathbf{u} = (\cos\theta, \sin\theta) \in \mathbb{R}^2$, where $-\pi \leq \theta \leq \pi$. (1) Compute the length of $\mathbf{u}$, (2) compute the vector $\mathbf{v}$ in the direction of $\mathbf{u}$ and $\parallel \mathbf{v} \parallel = 4$, (3) compute the vector $\mathbf{w}$ in the direction of opposite to $\mathbf{u}$ and same length.

Solution: (1) We have $\parallel \mathbf{u} \parallel = \sqrt{\cos^2\theta + \sin^2\theta} = 1$
(2) Length of $\mathbf{v}$ is four times that of $\mathbf{u}$, and they have same direction. So, $\mathbf{v} = 4\mathbf{u} = 4(\cos\theta, \sin\theta)$.
(3) $\mathbf{w} = -\mathbf{u} = -(\cos\theta, \sin\theta)$.

Example 6.1.4

Let **v** be a vector in the same direction as

$$\mathbf{u} = (-1, \pi, 1) \quad \textit{and} \quad \|\mathbf{v}\| = 4.$$

Compute **v**.

Solution: Write $\mathbf{v} = c\mathbf{u}$ with $c > 0$. Given $\|\mathbf{v}\| = 4$ So,

$$4 = \|\mathbf{v}\| = \|c\mathbf{u}\| = |c| \, \|\mathbf{u}\| = c\sqrt{(-1)^2 + \pi^2 + 1^2} = c\sqrt{\pi^2 + 2}$$

So, $c = \frac{4}{\sqrt{\pi^2+2}}$ and $\mathbf{v} = c\mathbf{u} = \frac{4}{\sqrt{\pi^2+2}}(-1, \pi, 1)$.

Example 6.1.5

Let $\mathbf{v} = (-1, 3, \sqrt{2}, \pi)$.

- (1) Find $\mathbf{u}$ such that $\mathbf{u}$ has same direction as $\mathbf{v}$ and one-half its length.
  Solution: In general,
  $$\| c\mathbf{v} \| = |c| \, \| \mathbf{v} \| \, .$$
  So, in this case,
  $$\mathbf{u} = \frac{1}{2}\mathbf{v} = \frac{1}{2}\left(-1, 3, \sqrt{2}, \pi\right) = \left(-\frac{1}{2}, \frac{3}{2}, \frac{1}{\sqrt{2}}, \frac{\pi}{2}\right).$$

Continued

- (2) Find **u** such that **u** has opposite direction as **v** and one-fourth its length.
  Solution: Since it has opposite direction

$$\mathbf{u} = -\frac{1}{4}\mathbf{v} = -\frac{1}{4}\left(-1, 3, \sqrt{2}, \pi\right) = \left(\frac{1}{4}, -\frac{3}{4}, -\frac{1}{2\sqrt{2}}, -\frac{\pi}{4}\right)$$

- (3) Find **u** such that **u** has opposite direction as **v** and twice its length.
  Solution: Since it has opposite direction

$$\mathbf{u} = -2\mathbf{v} = -2\left(-1, 3, \sqrt{2}, \pi\right) = (2, -6, -2\sqrt{2}, -2\pi).$$

Example 6.1.6

Find the distance between

$$\mathbf{u} = (-1, 2, 3, \pi) \qquad \textit{and} \qquad \mathbf{v} = (1, 0, 5, \pi + 2).$$

Solution: Distance

$$d(\mathbf{u}, \mathbf{v}) = \parallel \mathbf{u} - \mathbf{v} \parallel = \parallel (-2, 2, -2, -2) \parallel$$

$$= \sqrt{-(2)^2 + 2^2 + (-2)^2 + (-2)^2} = 4.$$

Definition: Dot Product

Definition. Let

$$\mathbf{u} = (u_1, u_2, \ldots, u_n), \quad \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$$

be two vectors in $\mathbb{R}^n$. The dot product of **u** and **v** is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Theorem 6.1.3

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are three vectors and c is a scalar. Then

1. (*Commutativity*): $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
2. (*Distributivity*): $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
3. (*Associativity*): $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$.
4. (*dot product and Norm*): $\mathbf{v} \cdot \mathbf{v} = \| \mathbf{v} \|^2$.
5. We have $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} \iff \mathbf{v} = \mathbf{0}$.

Proof. Follows from definition of dot product.

Remark. The vector space $\mathbb{R}^n$ together with (1) length, (2) dot product is called the Euclidean $n-$Space.

Theorem 6.1.4: Cauchy-Schwartz Inequality

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two vectors. Then,

$$|\mathbf{u} \cdot \mathbf{v}| \quad \leq \quad \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

Proof.

- (*Case 1.*): Assume $\mathbf{u} = \mathbf{0}$.
  - Then, $\|\mathbf{u}\| = 0$ and the Right Hand Side is zero.
  - Also, the Left Hand Side $= |\mathbf{u} \cdot \mathbf{v}| = |\mathbf{0} \cdot \mathbf{v}| = 0$
  - So, both sides are zero and the inequality is valid.

Continued

- (*Case 2.*): Assume $\mathbf{u} \neq \mathbf{0}$. So, $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 > 0$. Then,
  - Let t be any real number (variable) . We have
$$(t\mathbf{u} + \mathbf{v}) \cdot (t\mathbf{u} + \mathbf{v}) = \|(t\mathbf{u} + \mathbf{v})\|^2 \geq 0.$$
  - Expanding:
$$t^2(\mathbf{u} \cdot \mathbf{u}) + 2t(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{v}) \geq 0.$$
  - Write
$$a = \mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 > 0, \quad b = 2(\mathbf{u} \cdot \mathbf{v}), \quad c = (\mathbf{v} \cdot \mathbf{v}).$$
  - The above inequality can be written as
$$f(t) = at^2 + bt + c \geq 0 \qquad \textit{for all} \quad t.$$

Continued

- - From the graph of $y = f(t)$, we can see that, $f(t) = 0$ either has no real root or has a single repeated root.
  - By the Quadratic formula, we have

$$b^2 - 4ac \le 0 \qquad or \qquad b^2 \le 4ac.$$

  - This means

$$4(\mathbf{u} \cdot \mathbf{v})^2 \le 4(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) = 4 \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 .$$

  - Taking square root, we have

$$|\mathbf{u} \cdot \mathbf{v}| \quad \le \quad \|\mathbf{u}\| \|\mathbf{v}\| .$$

The proof is complete. ■

Definition: Angle Between Two Vectors

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two nonzero vectors.

- Cauchy-Swartz Inequality ensures $-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1$. So, the following definition makes sense.
- Definition. The angle θ between $\mathbf{u}, \mathbf{v} \in V$ is defined by the equation:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \qquad 0 \leq \theta \leq \pi.$$

- Definition We say that they are orthogonal, if $\mathbf{u}\cdot\mathbf{v} = 0$.

Theorem 6.1.5: Trianguler Inequality

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two vectors. Then,

$$\|\mathbf{u}+\mathbf{v}\| \quad \leq \quad \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Proof. First,

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2.$$

By Cauchy-Schwartz Inequality $|\ \mathbf{u}\cdot\mathbf{v}\ | \quad \leq \quad \|\ \mathbf{u}\ \|\|\ \mathbf{v}\ \|$. So,

$$\|\ \mathbf{u}+\mathbf{v}\ \|^2 \leq \|\ \mathbf{u}\ \|^2 + 2\,\|\ \mathbf{u}\ \|\|\ \mathbf{v}\ \| + \|\ \mathbf{v}\ \|^2 = (\|\ \mathbf{u}\ \| + \|\ \mathbf{v}\ \|)^2$$

The theorem is established by taking square root. ■

Theorem 6.1.6: Pythagorean

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two orthogonal vectors. Then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

Proof.

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}+2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2 .$$

The proof is complete. ∎

Example 6.1.7

Let $\mathbf{u} = (0, 1, -1, 1, -1)$ and $\mathbf{v} = (\sqrt{5}, 1, -3, 3, -1)$.

- (1) Find $\mathbf{u} \cdot \mathbf{v}$.
  Solution: We have

$$\mathbf{u} \cdot \mathbf{v} = (0, 1, -1, 1, -1) \cdot (\sqrt{5}, 1, -3, 3, -1)$$

$$= 0 + 1 + 3 + 3 + 1 = 8.$$

- (2) Compute $\mathbf{u} \cdot \mathbf{u}$.
  Solution: We have

$$\mathbf{u} \cdot \mathbf{u} = (0, 1, -1, 1, -1) \cdot (0, 1, -1, 1, -1) = 4$$

Continued

- (3) Compute $\| \mathbf{u} \|^2$.
  Solution: From (2), we have
  $$\| \mathbf{u} \|^2 = \mathbf{u} \cdot \mathbf{u} = 4.$$
- (4) Compute $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$.
  Solution: From (1), we have
  $$(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 4\mathbf{v} = 4(\sqrt{5}, 1, -3, 3, -1) = (4\sqrt{5}, 4, -12, 12, -4).$$

Example 6.1.7

Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^n$. It is given,

$$\mathbf{u} \cdot \mathbf{u} = 9, \quad \mathbf{u} \cdot \mathbf{v} = -7, \quad \mathbf{v} \cdot \mathbf{v} = 16.$$

Find $(3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v})$.

Solution: We have

$$(3\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-3\mathbf{v}) = 3\mathbf{u}\cdot\mathbf{u}-10\mathbf{u}\cdot\mathbf{v}+3\mathbf{v}\cdot\mathbf{v} = 3*9-10*(-7)+3*16 = 5$$

Example 6.1.8

Let $\mathbf{u} = (1, -\sqrt{2}, 1)$ and $\mathbf{v} = (2\sqrt{2}, 3, -2\sqrt{2})$. Verify Cauchy-Schwartz inequality.

Solution: We have

$$\|\mathbf{u}\| = \sqrt{1^2 + (-\sqrt{2})^2 + 1^2} = 2 \quad \textit{and}$$

$$\|\mathbf{v}\| = \sqrt{(2\sqrt{2})^2 + 3^2 + (-2\sqrt{2})^2} = 5.$$

$$\textit{Also} \quad \mathbf{u} \cdot \mathbf{v} = 1 * 2\sqrt{2} + (-\sqrt{2}) * (3), +1 * (-2\sqrt{2}) = -3\sqrt{2}.$$

Therefore, it is verified that

$$|\mathbf{u} \cdot \mathbf{v}| = |3\sqrt{3}| = 3\sqrt{2} \leq 2 * 5 = \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

Example 6.1.9

Let $\mathbf{u} = (1, -\sqrt{2}, 1)$ and $\mathbf{v} = (2\sqrt{2}, 0, -2\sqrt{2})$. Find the angle θ between them.

Solution: The **angle** θ between $\mathbf{u}$ and $\mathbf{v}$ is defined by the equation

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\parallel \mathbf{u} \parallel \parallel \mathbf{v} \parallel} \qquad 0 \leq \theta \leq \pi.$$

- We have

$$\|\mathbf{u}\| = \sqrt{1^2 + (-\sqrt{2})^2 + 1^2} = 2 \quad \textit{and}$$

$$\parallel \mathbf{v} \parallel = \sqrt{8 + 0 + 8} = 4$$

Continued

- Also
$$\mathbf{u} \cdot \mathbf{v} = 1 * \sqrt{2} + (-\sqrt{2} * 0 + 1 * (-2\sqrt{2}) = 0.$$
- So,
$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} = 0.$$
- Therefore,
$$\theta = \pi/2.$$

Example 6.1.10

Let $\mathbf{u} = (1, -3, -2, -7)$. Find all vectors that are orthogonal to $\mathbf{u}$.

Solution: Suppose $\mathbf{v} = (x_1, x_2, x_3, x_4)$ be orthogonal to $\mathbf{u}$. By definition, it means,

$$\mathbf{u} \cdot \mathbf{v} = x_1 - 3x_2 - x_3 - 7x_4 = 0$$

A parametric solution to this system is

$$x_2 = s, \ \ x_3 = t, \ \ x_4 = u, \ \ x_1 = 3s + 2t + 7u$$

So, the set of vectors orthogonal to $\mathbf{u}$, is given by

$$\{\mathbf{v} = (3s + 2t + 7u, s, t, u) : \quad s, t, u \in \mathbb{R}\}$$

Example 6.1.11

Let $\mathbf{u} = (\pi, 7, \pi)$ and $\mathbf{v} = (\sqrt{3}, 0, -\sqrt{3})$ Determine, if are $\mathbf{u}, \mathbf{v}$ orthogonal to each other or not?

Solution: We need to check, if $\mathbf{u} \cdot \mathbf{v} = 0$ or not. We have

$$\mathbf{u} \cdot \mathbf{v} = \pi * (\sqrt{3}) + 7 * 0 + \pi * (-\sqrt{3}) = 0$$

So, $\mathbf{u}, \mathbf{v}$ are orthogonal to each other.

Example 6.1.12

Let $\mathbf{u} = (\pi, 7, \pi)$ and $\mathbf{v} = (\sqrt{3}, 1, -\sqrt{3})$ Determine if are $\mathbf{u}, \mathbf{v}$ orthogonal to each other or not?

Solution: We need to check, if $\mathbf{u} \cdot \mathbf{v} = 0$ or not. We have

$$\mathbf{u} \cdot \mathbf{v} = \pi * (\sqrt{3}) + 7 * 1 + \pi * (-\sqrt{3}) = 7 \neq 0.$$

So, $\mathbf{u}, \mathbf{v}$ are not orthogonal to each other.

Example 6.1.13

Let $\mathbf{u} = (\sqrt{3}, \sqrt{3}, \sqrt{3})$, $\mathbf{v} = (-\sqrt{3}, -\sqrt{3}, -2\sqrt{3})$. Verify, triangle Inequality. **Solution:** We have

$$\|\mathbf{u}\| = \sqrt{(\sqrt{3})^2 + (\sqrt{3})^2 + (\sqrt{3})^2} = 3,$$

$$\|\mathbf{v}\| = \sqrt{(\sqrt{3})^2 + (-\sqrt{3})^2 + (-2\sqrt{3})^2} = 3\sqrt{2}$$

$$\|\mathbf{u} + \mathbf{v}\| = \left\|(0, 0, -\sqrt{3})\right\| = \sqrt{0^2 + 0^2 + (-\sqrt{3})^2} = \sqrt{3}.$$

$$\text{To Check :} \quad \|\mathbf{u} + \mathbf{v}\|^2 = 3 \quad \leq \quad \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 9 + 18.$$

So, the triangle inequality is verified.

Example 6.1.14

Let $\mathbf{u} = (1, -1)$, $\mathbf{v} = (2, 2)$. Verify Pythagorean Theorem.

Solution:

- We have $\mathbf{u} \cdot \mathbf{v} = 1 * 2 - 1 * 2 = 0$. So, $\mathbf{u}, \mathbf{v}$ are orthogonal to each other and Pythagorean Theorem must hold.
-
$$\parallel \mathbf{u} \parallel = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \quad \parallel \mathbf{v} \parallel = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$
$$\parallel \mathbf{u} + \mathbf{v} \parallel = \parallel (3, 1) \parallel = \sqrt{3^2 +^2} = \sqrt{10}.$$
- We need to check,
$$\parallel \mathbf{u} + \mathbf{v} \parallel^2 = 10 = \parallel \mathbf{u} \parallel^2 + \parallel \mathbf{v} \parallel^2 = 2 + 8$$
- So, the Pythagorian Theorem is verified.

Example 6.1.15

Let $\mathbf{u} = (a, b)$, $\mathbf{v} = (b, -a)$. Verify Pythagorean Theorem.
Solution:

- We have $\mathbf{u} \cdot \mathbf{v} = ab - ba = 0$. So, $\mathbf{u}, \mathbf{v}$ are orthogonal to each other and Pythagorean Theorem must hold.
- $$\| \mathbf{u} \| = \sqrt{a^2 + b^2}, \quad \| \mathbf{v} \| = \sqrt{b^2 + a^2}$$
  $$\| \mathbf{u} + \mathbf{v} \| = \| (a + b, b - a) \|$$
  $$= \sqrt{(a + b)^2 + (b - a)^2} = \sqrt{2(a^2 + b^2)}$$

Continued

- We need to check,

$$\| \mathbf{u}+\mathbf{v} \|^2 = 2(a^2+b^2) = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 = (a^2+b^2)+(b^2+a^2)$$

- So, the Pythagorean Theorem is verified.