Chapter 7: Linear Transformations
§ 7.2 Properties of Homomorphisms

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Goals

- In this section we discuss the fundamental properties of homomorphisms of vector spaces.
- Reminder: We remind ourselves that homomorphisms of vectors spaces are also called Linear Maps and Linear Transformations. We use these three expressions, inter changeably.

Lemma 7.2.1:The First Property

Property: Suppose V, W are two vector spaces and $T : V \longrightarrow W$ is a homomorphism. Then, $T(\mathbf{0}_V) = \mathbf{0}_W$, where $\mathbf{0}_V$ denotes the zero of V and $\mathbf{0}_W$ denotes the zero of W.

(**Notations**: *When clear from the context, to denote zero of the respective vector space by* $\mathbf{0}$*; and drop the subscript* V, W *etc.*)

Continued

Proof. We have $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$. Add $-T(\mathbf{0})$ on both sides of the equation. We have

$$T(\mathbf{0}) - T(\mathbf{0}) = (T(\mathbf{0}) + T(\mathbf{0})) - T(\mathbf{0})$$

$$\text{So,} \quad \mathbf{0}_W = T(\mathbf{0}) + (T(\mathbf{0})) - T(\mathbf{0}) = T(\mathbf{0}) + \mathbf{0}_W = T(\mathbf{0}).$$

Theorem 7.2.2:Equivalent Characterization

Theorem 7.2.2: Suppose V, W are two vector spaces and $T: V \longrightarrow W$ is a function (set theoretic). Then, T is a homomorphism (i. e. Linear map) if and only if

$$T(r\mathbf{u}+s\mathbf{v}) = rT(\mathbf{u})+sT(\mathbf{v}) \quad \text{for all } \mathbf{u},\mathbf{v} \in V \;\; r,s \in \mathbb{R}. \quad (1)$$

Proof. Suppose the condition (1) holds. We will prove that T is a homomorphism. First, with $r = s = 1$, it follows from condition (1)

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v}) \quad \text{for all } \mathbf{u},\mathbf{v} \in V.$$

Continued

Taking $s = 0$, it follows from condition (1)

$$T(r\mathbf{u}) = T(r\mathbf{u} + 0\mathbf{v}) = rT(\mathbf{u}) + 0T(\mathbf{v}) = rT(\mathbf{u})$$

So, it is established that T is a homomorphism.
Conversely, suppose T is a homomorphism. We will prove that condition (1) holds. From the first, then second property of homomorphism, it follows

$$T(r\mathbf{u} + s\mathbf{v}) = T(r\mathbf{u}) + T(s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v})$$

So, the equation (1) is established.

Corollary 7.2.3:Linearity, with finite sum

Suppose V, W are two vector spaces and $T : V \longrightarrow W$ is a homomorphism (i. e. Linear map). Let $\mathbf{u}_1, \ldots, \mathbf{u}_n \in V$ be n vectors and $c_1, \ldots, c_n \in \mathbb{R}$ be n scalars. Then,

$$T(c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n) = c_1 T(\mathbf{u}_1) + \cdots + c_n T(\mathbf{u}_n).$$

Proof. (*We use method of induction, to prove this.*) The method has two steps.

Continued

- (**Initialization Step**:) When, $n = 1$, we need to prove $T(c_1\mathbf{u}_1) = c_1T(\mathbf{u}_1)$. This follows from the second condition of the definition homomorphisms.
- (**Induction Step**:) We assume the the proposition is valid for fewer than $n-1$ summands and prove it for n summands.
  By this assumption
  $$T(c_1\mathbf{u}_1 + \cdots + c_{n-1}\mathbf{u}_{n-1}) = c_1T(\mathbf{u}_1) + \cdots + c_{n-1}T(\mathbf{u}_{n-1}).$$
  $$\text{By n} = 1 \text{ case}: \qquad T(c_n\mathbf{u}_n) = c_nT(\mathbf{u}_n)$$

Continued

- By first condition of the definition homomorphisms

$$T\left(c_1\mathbf{u}_1+\cdots+c_{n-1}\mathbf{u}_{n-1}+c_n\mathbf{u}_n\right)$$

$$=T\left(c_1\mathbf{u}_1+\cdots+c_{n-1}\mathbf{u}_{n-1}\right)+T\left(c_n\mathbf{u}_n\right)$$

$$=c_1T\left(\mathbf{u}_1\right)+\cdots+c_{n-1}T\left(\mathbf{u}_{n-1}\right)+c_nT\left(\mathbf{u}_n\right)$$

The proof is complete.

Prelude.

A basis of a vector space V dictates most of the properties of V. The next theorem does exactly the same for Homomorphisms (i. e. linear maps) $T : V \longrightarrow W$.

Theorem 7.2.4:Bases and Linear Maps

Let V, W be two vector spaces. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a basis of V and $\mathbf{w}_1, \ldots, \mathbf{w}_m \in W$ be m vectors in W. Then,

- (**Existence**): Then, there is a homomorphisms $T : V \longrightarrow W$ such that
$$T(\mathbf{v}_1) = \mathbf{w}_1, \ T(\mathbf{v}_2) = \mathbf{w}_2, \ \cdots, \ T(\mathbf{v}_m) = \mathbf{w}_m. \quad (2)$$
- (**Uniqueness**): The Equation(2) determines a unique homomorphisms $T : V \longrightarrow W$.

Continued

Proof.

- First, we define $T : V \longrightarrow W$. Let $\mathbf{x} \in V$. By property of bases, there are scalars $c_1, \ldots, c_n \in \mathbb{R}$, such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_m$$

$$\text{Define} \quad T(\mathbf{x}) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_m$$

Since, for a given $\mathbf{x} \in V$, $c_1, \ldots, c_m$ are unique (*there were no choices involved*), the right hand side depends only on $\mathbf{x}$. So, $T(\mathbf{x})$ is well defined.

(*One needs to justify so called "well defined-ness", whenever something is defined.*)

Continued

To complete the proof of existence, we need to show T is a homomorphism. Let $\mathbf{x} \in V$ be as above and $\mathbf{y} \in V$ be another vector. There are scalars $d_1, \ldots, d_n \in \mathbb{R}$, such that

$$\left\{ \begin{array}{l} \mathbf{y} = d_1\mathbf{v}_1 + \cdots + d_m\mathbf{v}_m. \text{ By definition,} \\ T(\mathbf{y}) = d_1\mathbf{w}_1 + \cdots + d_m\mathbf{w}_m \\ \mathbf{x} + \mathbf{y} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_m + d_m)\mathbf{v}_m \\ \text{ByDefinition, } T(\mathbf{x} + \mathbf{y}) = (c_1 + d_1)\mathbf{w}_1 + \cdots + (c_n + d_n)\mathbf{w}_n \\ = (c_1\mathbf{w}_1 + + \cdots + c_m\mathbf{w}_m) + (d_1\mathbf{w}_1 + \cdots + d_m\mathbf{w}_m) \\ = T(\mathbf{x}) + T(\mathbf{y}). \end{array} \right.$$

The first (the additive) property of homomorphism is checked.

Continued

Now, we prove the T satisfies the second condition:

- Let $\mathbf{x} \in V$ be as above and $r \in \mathbb{R}$ be a scalars. Then,

$$\begin{cases} r\mathbf{x} = (rc_1)\mathbf{v}_1 + \cdots + (rc_n)\mathbf{v}_m \\ \text{By Definition, } T(r\mathbf{x}) = (rc_1)\mathbf{w}_1 + \cdots + (rc_m)\mathbf{w}_m \\ = (rc_1)\mathbf{w}_1 + + \cdots + (rc_m)\mathbf{w}_m = r\left(c_1\mathbf{w}_1 + + \cdots + c_m\mathbf{w}_m\right) \\ = rT(\mathbf{x}) \end{cases}$$

The second property of homomorphism is checked. So, it is established that T is a homomorphism (i. e. Existence part).

Continued

Now we prove uniqueness part. Let $T_1, T_2 : V \longrightarrow W$ be two homomorphism, satisfying Equation (2), meaning

$$\begin{cases} T_1(\mathbf{v}_1) = \mathbf{w}_1 & T_1(\mathbf{v}_2) = \mathbf{w}_2, & \cdots, & T_1(\mathbf{v}_m) = \mathbf{w}_m \\ T_2(\mathbf{v}_1) = \mathbf{w}_1 & T_2(\mathbf{v}_2) = \mathbf{w}_2, & \cdots, & T_2(\mathbf{v}_m) = \mathbf{w}_m \end{cases}$$

For all $\mathbf{x} \in V$, we need to prove $T_1(\mathbf{x}) = T_2(\mathbf{x})$. As before $\mathbf{x} \in V$, we can write

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \quad \text{for some } c_1, \ldots, c_m \in \mathbb{R}.$$

Continued

Since $T_1, T_2 : V \longrightarrow W$ are homomorphism,

$$\begin{cases} T_1(\mathbf{x}) = T_1(\mathbf{v}_1) + c_2 T_1(\mathbf{v}_2) + \cdots + c_m T_1(\mathbf{v}_m) \\ = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_m\mathbf{w}_m \\ \text{Likewise,} \\ T_2(\mathbf{x}) = T_2(\mathbf{v}_1) + c_2 T_2(\mathbf{v}_2) + \cdots + c_m T_2(\mathbf{v}_m) \\ = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_m\mathbf{w}_m \end{cases}$$

So, $T_1(\mathbf{x}) = T_2(\mathbf{x})$. This completes the proof of Part 2 (uniqueness part).

Example 7.2.1

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \qquad (3)$$

be the standard basis of $\mathbb{R}^3$. Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

Then, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ forms a basis of $\mathbb{R}^3$ (*we do not prove this*)

Continued

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the homomorphism defined by

$$T(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad T(\mathbf{v}_3) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

Compute $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)$. More generally, compute $T\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Continued

Solution: We will write $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. First, write $\mathbf{e}_1 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$. In matrix form:

$$\mathbf{e}_1 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

Continued

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1}\mathbf{e}_1 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

$$\text{So, } \mathbf{e}_1 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}\begin{pmatrix} .5 \\ -.5 \\ 0 \end{pmatrix} = .5(\mathbf{v}_1 - \mathbf{v}_2)$$

$$\text{So } T(\mathbf{e}_1) = .5\,(T(\mathbf{v}_1) - T(\mathbf{v}_2)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Continued

Now compute $T(\mathbf{e}_2)$. As before , write $\mathbf{e}_2 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$

$$\text{So } \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1}\mathbf{e}_2 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix}$$

$$\text{So, } \mathbf{e}_2 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}\begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix} = .25\mathbf{v}_1 + .25\mathbf{v}_2 - .5\mathbf{v}_3$$

$$\text{So } T(\mathbf{e}_2) = .25\left(T(\mathbf{v}_1) + .25\,T(\mathbf{v}_2)\right) - .5\,T(\mathbf{v}_3)$$

$$= .25\begin{pmatrix} 1 \\ 1 \end{pmatrix} + .25\begin{pmatrix} -1 \\ 1 \end{pmatrix} - .5\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Continued

Now compute $T(\mathbf{e}_3)$. As before , write $\mathbf{e}_3 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$

$$\text{So} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1}\mathbf{e}_3 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} .25 \\ .25 \\ .5 \end{pmatrix}$$

$$\text{So, } \mathbf{e}_3 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}\begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix} = .25\mathbf{v}_1 + .25\mathbf{v}_2) + 5\mathbf{v}_3$$

$$\text{So } T(\mathbf{e}_3) = .25\,(T(\mathbf{v}_1) + .25\,T(\mathbf{v}_2)) + .5\,T(\mathbf{v}_3)$$

$$= .25\begin{pmatrix} 1 \\ 1 \end{pmatrix} + .25\begin{pmatrix} -1 \\ 1 \end{pmatrix} + .5\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Continued

Finally,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3.$$

So,

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = aT(\mathbf{e}_1) + bT(\mathbf{e}_2) + cT(\mathbf{e}_3)$$

$$= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Example 7.2.2

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ be the standard basis of $\mathbb{R}^3$, as in (3) and Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

Then, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ forms a basis of $\mathbb{R}^3$ (*we do not prove this*)
Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the homomorphism defined by

$$T(\mathbf{v}_1) = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad T(\mathbf{v}_3) = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

Continued

Compute $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)$. More generally, compute $T\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Solution: Steps are very similar to the above problem. We will write $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. First, write $\mathbf{e}_1 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$.

Continued

In matrix form:

$$\mathbf{e}_1 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

Continued

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1}\mathbf{e}_1 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

$$\text{So, } \mathbf{e}_1 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}\begin{pmatrix} .5 \\ -.5 \\ 0 \end{pmatrix} = .5(\mathbf{v}_1 - \mathbf{v}_2)$$

Continued

$$\text{So } T(\mathbf{e}_1) = .5\,(T(\mathbf{v}_1) - T(\mathbf{v}_2))$$

$$= .5\begin{pmatrix} -1 \\ -1 \\ -1 \\ 3 \end{pmatrix} - .5\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Continued

Now compute $T(\mathbf{e}_2)$. As before , write $\mathbf{e}_2 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$

$$\text{So } \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1}\mathbf{e}_2 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix}$$

$$\text{So, } \mathbf{e}_2 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}\begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix} = .25\mathbf{v}_1 + .25\mathbf{v}_2 - .5\mathbf{v}_3$$

Continued

$$\text{So } T(\mathbf{e}_2) = .25\,(T(\mathbf{v}_1) + .25\,T(\mathbf{v}_2)) - .5\,T(\mathbf{v}_3)$$

$$= .25 \begin{pmatrix} -1 \\ -1 \\ -1 \\ 3 \end{pmatrix} + .25 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} - .5 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Continued

Now compute $T(\mathbf{e}_3)$. As before , write $\mathbf{e}_3 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$

$$\text{So } \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1}\mathbf{e}_3 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} .25 \\ .25 \\ .5 \end{pmatrix}$$

$$\text{So, } \mathbf{e}_3 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}\begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix} = .25\mathbf{v}_1 + .25\mathbf{v}_2 + 5\mathbf{v}_3$$

Continued

$$\text{So } T(\mathbf{e}_3) = .25T(\mathbf{v}_1) + .25T(\mathbf{v}_2) + .5T(\mathbf{v}_3)$$

$$= .25\begin{pmatrix} -1 \\ -1 \\ -1 \\ 3 \end{pmatrix} + .25\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + .5\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Continued

$$\text{Finally,} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3.$$

$$\text{So,} \quad T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = aT(\mathbf{e}_1) + bT(\mathbf{e}_2) + cT(\mathbf{e}_3)$$

$$= a\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \\ a+b+c \end{pmatrix}$$

Exercises 1

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ be the standard basis of $\mathbb{R}^3$, as in (3)

$$\text{and}\quad \mathbf{v}_1 = \begin{pmatrix} \frac{1}{2} \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -1 \end{pmatrix} \in \mathbb{R}^3$$

Then, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis of $\mathbb{R}^3$ (*need not check*). Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be be the homomorphism, defined by

$$T(\mathbf{v}_1) = \mathbf{v}_1, \quad T(\mathbf{v}_2) = \mathbf{0}, \quad T(\mathbf{v}_3) = \mathbf{0}. \quad \text{Compute}$$

$T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)$, and in particular $T\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Exercise 2

Let $\mathbb{P}_2(\mathbb{R})$ be the vector space of all polynomials f with $\deg(f) \leq 2$. Then,

$$\mathbf{p}_1(x) = 1+x+x^2, \ \ \mathbf{p}_2(x) = x+x^2, \mathbf{p}_3(x) = x^2 \text{ is a basis of } \mathbb{P}_2(\mathbb{R}).$$

Define the homomorphism $T : \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{M}_2(\mathbb{R})$ by

$$T(\mathbf{p}_1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \ T(\mathbf{p}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ T(\mathbf{p}_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Compute $T(1), T(x), T(x^2)$ and in general $T(a + bx + cx^2)$.

Preview

Recall, given a matrix $A \in \mathbb{M}_{m\times n}(\mathbb{R})$, there is a an homomorphism

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{defined by } T(\mathbf{x}) = A\mathbf{x} \quad \forall\, \mathbf{x} \in \mathbb{R}^n.$$

We demonstrate that, any homomorphism $T : V \longrightarrow W$ of vectors spaces, with finite dimension, are determined by matrices.

Theorem 7.2.5:Matrices to Homomorphisms

Let V, W be two vector spaces.

- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$ be a basis of V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \in W$ be elements in W.
- Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{M}_{m\times n}(\mathbb{R})$$

Continues:Matrix A to T_A

Then, there is a homomorphism $T_A : V \longrightarrow W$ such that

$$\begin{cases} T_A(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{12}\mathbf{w}_2 + \cdots + a_{1n}\mathbf{w}_n \\ T_A(\mathbf{v}_2) = a_{21}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{2n}\mathbf{w}_n \\ \cdots \\ T_A(\mathbf{v}_m) = a_{m1}\mathbf{w}_1 + a_{m2}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_n \end{cases} \qquad (4)$$

In matrix notation,

$$\begin{pmatrix} T_A(\mathbf{v}_1) \\ T_A(\mathbf{v}_2) \\ \cdots \\ T_A(\mathbf{v}_m) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \cdots \\ \mathbf{w}_n \end{pmatrix} \qquad (5)$$

Continues

Proof. It follows from Theorem 7.2.4 and equation (4) and theorem above. ■

We Remark:

- The notation T_A was chosen, with subscript A, to show its dependence on A, and for future reference.
- T_A, also, depend on the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m \in V\}$ and elements $\{\mathbf{w}_1, \ldots, \mathbf{w}_m \in W\}$. That means, if we change the basis $\mathbf{v}_i$, or elements $\{\mathbf{w}_j\}$ the homomorphism T_A we get will be different.

Continues

- Suppose $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m \in V\}$ be the standard basis of $V = \mathbb{R}^m$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m \in W\}$ be the standard basis of $W = \mathbb{R}^n$ (as in Equation 3). Then,
$$T_A(\mathbf{x}) = A^t\mathbf{x} \qquad \forall\ \mathbf{x} \in \mathbb{R}^m.$$
This example was discussed before.
- A converse of the above is also valid as follows.

Theorem 7.2.6:Homomorphisms to Matrices

Let V, W be two vector spaces. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$ be a basis of V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \in W$ be basis in W.
Let $T : V \longrightarrow W$ be a homomorphism. Since, $\{\mathbf{w}_j\}$ is a basis of W, we can back track the above steps and write uniquely:

$$\begin{cases} T(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{12}\mathbf{w}_2 + \cdots + a_{1n}\mathbf{w}_n \\ T(\mathbf{v}_2) = a_{21}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{2n}\mathbf{w}_n \\ \cdots \\ T(\mathbf{v}_m) = a_{m1}\mathbf{w}_1 + a_{m2}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_n \end{cases} \quad (6)$$

with unique $a_{ij} \in \mathbb{R}$.

Continued: T to Matrix

This way we get a well defined matrix

$$\mathbf{A}_T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{7}$$

We remark, the $\mathbf{A}_T$ depends on the choice of bases of V and of W, as above.

Theorem 7.2.7:The Correspondence

Let V, W be two vector spaces. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be a basis of V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ be basis in W. Let $\mathcal{L}(V, W)$ be the set of all homomorphisms $V \longrightarrow W$.

- Define $\varphi : \mathcal{L}(V, W) \longrightarrow \mathbb{M}_{m \times n}(\mathbb{R})$ by $\varphi(T) = A_T$, where $A_T \in \mathbb{M}_{m \times n}(\mathbb{R})$ is the matrix is as in Theorem 7.2.6. Then, φ is a well defined bijective correspondence.
- Define $\psi : \mathbb{M}_{m \times n}(\mathbb{R}) \longrightarrow \mathcal{L}(V, W)$ by $\psi(A) = T_A$, where $T_A \in \mathcal{L}(V, W)$ is as in Theorem 7.2.5. Then, ψ and φ are the inverses of each other.

Proof

Proof. It follows from the above discussions on the definitions of A_T and T_A. We skip the details of the proof. ■

Remark. *We comment that A_T and T_A depend on the choices of bases $\{\mathbf{v}_i\}$ of V and $\{\mathbf{w}_j\}$ of W. Hence φ and ψ would also do the same.*

Definitions and Theorem 7.2.8

Definitions. Let V, W be two vector spaces and $T : V \longrightarrow W$ is a homomorphism. Then, define

$$\begin{cases} \mathcal{N}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}. \\ \mathcal{R}(T) = \{\mathbf{w} \in W : w = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}. \end{cases}$$

- Then, $\mathcal{N}(T)$ is a subspace of V. This subspace $\mathcal{N}(T)$ is called the Null Space of T.
- Then, $\mathcal{R}(T)$ is a subspace of W. This subspace $\mathcal{R}(T)$ is called the Range of T.

Proof. Skip

Continued

As before $T : V \longrightarrow W$ be a homomorphism. Also, define

- *Nullity*$(T) = \dim \mathcal{N}(T)$.
- *rank*$(T) = \dim \mathcal{R}(T)$

Motivating Example

Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ and $T = T_A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be as above.

- Then, $\mathcal{N}(T) = \mathcal{N}(A)$. In words, the Null space of T and Null space of A are same.
  Therefore, $Nullity(T) = Nullity(A^t)$.
- Also, the range $\mathcal{R}(T)$ is equal to the column space of A.
  Therefore, $rank(T) = rank(A)$.

Theorem 7.2.9:Injective Homomorphisms

Let V, W be vector spaces and $T : V \longrightarrow W$ be a homomorphism. Then, T is injective if and only of the null space $\mathcal{N}(T) = \{\mathbf{0}\}$.

Proof. $(\Longrightarrow)$: Suppose T is injective and $\mathbf{x} \in \mathcal{N}(T)$. So, $T(\mathbf{x}) = \mathbf{0}_W = T(\mathbf{0})$. By injectivity of T, $\mathbf{x} = \mathbf{0}$. So, $\mathcal{N}(T) \subseteq \{\mathbf{0}\}$. So, $\mathcal{N}(T) = \{\mathbf{0}\}$
$(\Longleftarrow)$: Suppose $\mathcal{N}(T) = \{\mathbf{0}\}$. Let $\mathbf{x}_1, \mathbf{x}_2 \in V$ and $T(\mathbf{x}_1) = T(\mathbf{x}_2)$. Then, $T(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}_W$. So, $\mathbf{x}_1 - \mathbf{x}_2 \in \mathcal{N}(T) = \{\mathbf{0}\}$. So, $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$ and $\mathbf{x}_1 = \mathbf{x}_2$. So, T is injective. ■

Theorem 7.2.10:Bijective Homomorphisms

Let V, W be a vector spaces and $T : V \longrightarrow W$ be homomorphism. Then, the following three statements are equivalent.

1. T is bijective.
2. The null space $\mathcal{N}(T) = \{\mathbf{0}\}$ and range $\mathcal{R}(T) = W$.
3. $Nullity(T) = \{\mathbf{0}\}$ and range $\mathcal{R}(T) = W$.

Proof. Follows from the above. ∎

Isomorphisms

Defintion. Let V, W be a vector spaces. A bijective homomorphism $T : V \longrightarrow W$ is also called isomorphism. When there is such an isomorphism, we say V and W are isomorphic.

Theorem. Let V, W be a vector spaces and $T : V \longrightarrow W$ is an isomorphism. Let $G : W \longrightarrow V$ be the set theoretic inverse of T. Then, G is also an isomorphism.

Proof. Skip.

Isomorphisms: Remarks

If V and W, are isomorphic, then properties (Vector-space related) *of V translates to properties of W, and conversely. So, they can be treated as "same".* For Example:
Suppose $T : V \longrightarrow W$ is an isomorphism.

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V. Then, $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is a basis of W.
- So, $\dim V = \dim W$.

Lemma 7.2.11 (auxilery)

Let V be a vector space and $W \subseteq V$ be a subspace of V. Assume $\dim V < \infty$. Then, $W = V$ if and only if $\dim W = \dim V$. In particular, if V and W are isomorphic, then $\dim V = \dim W$.

Proof. It is obvious, if $W = V$ then, $\dim W = \dim V$. Now, assume $\dim W = \dim V = m$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be a basis of W. If $W \neq V$, there is $\mathbf{v} \in V$ and $\mathbf{v} \notin W$. Then, $\mathbf{w}_1, \ldots, \mathbf{w}_m, \mathbf{v}$ are linearly independent. This contradicts that $\dim V = m$. So, $V = W$. ■

Theorem 7.2.12:Isomorphism and Dimension

Let V, W be vector spaces, with $\dim V < \infty, \dim W < \infty$. Then, V and W are isomorphic if and only if $\dim V = \dim W$.

Proof. As was established above, if V, W are isomorphic then $\dim V = \dim W$. Now, suppose $\dim V = \dim W = n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V, and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is a basis of W. Let $T : V \longrightarrow W$ be the homomorphism, such that

$$T(\mathbf{v}_1) = \mathbf{w}_1, \ldots, T(\mathbf{v}_n) = \mathbf{w}_n$$

It is easy to see T is an isomorphism. ■

Corollary 7.2.13:Isomorphisms with $\mathbb{R}^n$

Suppose V is a vector space with $\dim V = n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis basis of $\mathbb{R}^n$. Then, the homomorphism $f : \mathbb{R}^n \longrightarrow V$, determined by,

$$f(\mathbf{e}_1) = \mathbf{v}_1, \ldots, f(\mathbf{e}_n) = \mathbf{v}_n$$

is an isomorphism. We would call this isomorphism the standard isomorphism.

Theorem 7.2.14:Nullity-Rank Theorem

Theorem. Let V, W be vector spaces, with $\dim V = m < \infty, \dim W = n < \infty$. Let $T : V \longrightarrow W$ be a homomorphism. Then,

$$Nullity(T) + rank(T) = \dim V = m.$$

Proof. Fix a basis $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of V and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of W. Let $A := A_T \in \mathbb{M}_{m\times n}(\mathbb{R})$ be the matrix of T, with respect to these bases. Let $A^t \in \mathbb{M}_{n\times m}(\mathbb{R})$ denote the transpose of A.

Continues

Close inspection shows, the diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{T_{A^t}} & \mathbb{R}^n \\ f\Big\downarrow\wr & & \wr\Big\downarrow g \\ V & \xrightarrow[T]{} & W \end{array} \qquad \text{commutes,}$$

where f, g are the standard isomorphisms. The restrictions of f establishes an isomorphism $f_0 : \mathcal{N}(T_{A^t}) \longrightarrow \mathcal{N}(T)$. So,

$$Nullity(A^t) = \dim \mathcal{N}(T_{A^t}) = \dim \mathcal{N}(T) = Nullity(T).$$

Continued

Likewise, restrictions of g establishes an isomorphism $g_0 : \mathcal{R}(T_{A^t}) \longrightarrow \mathcal{R}(T)$. So,

$$rank(A^t) = \dim \mathcal{R}(T_{A^t}) = \dim \mathcal{R}(T) = rank(T).$$

Recall, we proved

$$Nullity(A^t) + rank(A^t) = m \quad \text{(no of columns of) } \mathrm{A^t}$$

So,

$$Nullity(T) + rank(T) = \dim V = m.$$

Re-Define: Eigenvalues and Eigenvectors

We worked with eigenvalues and eigenvectors for matrices. Now, vector spaces V and linear transformations $T : V \longrightarrow V$, we define eigenvalues and eigenvectors. Let V be a vector space and $T : V \longrightarrow V$ be a Linear Transformation. A scalar $\lambda \in \mathbb{R}$ is said to be a eigenvalue of T, if $T(\mathbf{x}) = \lambda \mathbf{x}$ for some $\mathbf{x} \in V$, with $\mathbf{x} \neq \mathbf{0}$. In this case, $\mathbf{x}$ would be called eigenvector, of T, corresponding to λ.

Theorem 7.2.15:Eigen Value in Two Ways

Let V be a vector space and $T : V \longrightarrow V$ be a Linear Transformation. Assume $\dim V = n < \infty$. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V. Let $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ be the matrix of T, with respect to the basis B, on two sides. That means,

$$\begin{pmatrix} T(\mathbf{v}_1) \\ T(\mathbf{v}_2) \\ \cdots \\ T(\mathbf{v}_n) \end{pmatrix} = A \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \cdots \\ \mathbf{v}_n \end{pmatrix}$$

Continued

Also, let $f : \mathbb{R}^n \longrightarrow V$ be the standard isomorphism, and $g : V \longrightarrow \mathbb{R}^n$ be the inverse of f. So, f, g are determined by

$$\begin{cases} f(\mathbf{e}_1) = \mathbf{v}_1, \ldots, f(\mathbf{e}_n) = \mathbf{v}_n \\ g(\mathbf{v}_1) = \mathbf{e}_1, \ldots, g(\mathbf{v}_n) = \mathbf{e}_n \end{cases} \qquad \text{Then, for} \quad \lambda \in \mathbb{R}$$

following three conditions are equivalent:

- $\lambda \in \mathbb{R}$ is an eigen value of T
- $\lambda \in \mathbb{R}$ is an eigen value of A.
- $\lambda \in \mathbb{R}$ is an eigen value of A^t.

Continued

Further, corresponding an eigenvalue λ of T, $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of A^t if and only if $f(\mathbf{x})$ is an eigenvector of T.

Proof. Proof follows from the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{T_{A^t}} & \mathbb{R}^n \\ f \downarrow \wr & & \wr \downarrow f \\ V & \xrightarrow[T]{} & V \end{array}$$

More explicitly,

$$A^t\mathbf{x} = \lambda\mathbf{x} \Longleftrightarrow f\left(A^t\mathbf{x}\right) = f\left(\lambda\mathbf{x}\right) \Longleftrightarrow T\left(f(\mathbf{x})\right) = \lambda f(\mathbf{x}).$$

The proof is complete. ■

Theorem 7.2.16:Change of Basis

Let V be a vector space, with $\dim V = n$ and $T : V \longrightarrow V$ be a linear transformation. Let $\mathcal{B}_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $\mathcal{B}_2 = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ two bases of V.
Since both $\mathcal{B}_1$ and $\mathcal{B}_2$ are both bases, the is an invertible matrix P, expressing $\mathcal{B}_1$ in terms of $\mathcal{B}_2$, as follows:

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \cdots \\ \mathbf{v}_n \end{pmatrix} = P \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \cdots \\ \mathbf{w}_n \end{pmatrix} \tag{8}$$

Continued

- Using the basis $\mathcal{B}_1$ (*respectively.* $\mathcal{B}_2$) , for both domain and codomain, we have A_T, B_T, as follows:

$$\begin{pmatrix} T(\mathbf{v}_1) \\ T(\mathbf{v}_2) \\ \cdots \\ T(\mathbf{v}_n) \end{pmatrix} = A_T \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \cdots \\ \mathbf{v}_n \end{pmatrix}, \begin{pmatrix} T(\mathbf{w}_1) \\ T(\mathbf{w}_2) \\ \cdots \\ T(\mathbf{w}_n) \end{pmatrix} = B_T \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \cdots \\ \mathbf{w}_n \end{pmatrix} \quad (9)$$

Continued

These three matrices are related as follows:

$$B_T = P^{-1} A_T P$$

This is called the Change of Basis Formula.
Proof. Rewrite the first equation (9):

$$T \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \cdots \\ \mathbf{v}_n \end{pmatrix} = A_T \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \cdots \\ \mathbf{v}_n \end{pmatrix}$$

Continued

Using the equation (8), in this,

$$T\left(P\begin{pmatrix}\mathbf{w}_1\\ \mathbf{w}_2\\ \cdots\\ \mathbf{w}_n\end{pmatrix}\right)=A_T P\begin{pmatrix}\mathbf{w}_1\\ \mathbf{w}_2\\ \cdots\\ w_n\end{pmatrix}$$

$$\text{So,}\quad PT\begin{pmatrix}\mathbf{w}_1\\ \mathbf{w}_2\\ \cdots\\ \mathbf{w}_n\end{pmatrix}=A_T P\begin{pmatrix}\mathbf{w}_1\\ \mathbf{w}_2\\ \cdots\\ w_n\end{pmatrix}$$

Continued

$$\text{So,} \quad T\begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \cdots \\ \mathbf{w}_n \end{pmatrix} = P^{-1}A_TP\begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \cdots \\ w_n \end{pmatrix}$$

Comparing the second equation in (9), we have

$$B_T = P^{-1}A_TP.$$

This establishes the Change of Basis Formula. ■