Chapter 7: Linear Transformations

§ 7.2 Properties of Homomorphisms

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Summer 2017
In this section we discuss the fundamental properties of homomorphisms of vector spaces.

Reminder: We remind ourselves that homomorphisms of vectors spaces are also called Linear Maps and Linear Transformations. We use these three expressions, interchangeably.
Lemma 7.2.1: The First Property

**Property**: Suppose $V$, $W$ are two vector spaces and $T : V \longrightarrow W$ is a homomorphism. Then, $T(0_V) = 0_W$, where $0_V$ denotes the zero of $V$ and $0_W$ denotes the zero of $W$.

*(Notations: When clear from the context, to denote zero of the respective vector space by $0$; and drop the subscript $V$, $W$ etc.)*
**Proof.** We have \( T(0) = T(0 + 0) = T(0) + T(0) \).

Add \(-T(0)\) on both sides of the equation. We have

\[
T(0) - T(0) = (T(0) + T(0)) - T(0)
\]

So,

\[
0_W = T(0) + (T(0)) - T(0) = T(0) + 0_W = T(0).
\]
Theorem 7.2.2: Equivalent Characterization

Theorem 7.2.2: Suppose $V$, $W$ are two vector spaces and $T : V \rightarrow W$ is a function (set theoretic). Then, $T$ is a homomorphism (i.e. Linear map) if and only if

$$T(ru + sv) = rT(u) + sT(v) \quad \text{for all } u, v \in V, r, s \in \mathbb{R}. \quad (1)$$

Proof. Suppose the condition (1) holds. We will prove that $T$ is a homomorphism. First, with $r = s = 1$, it follows from condition (1)

$$T(u + v) = T(u) + T(v) \quad \text{for all } u, v \in V.$$
Taking \( s = 0 \), it follows from condition (1)

\[
T(ru) = T(ru + 0v) = rT(u) + 0T(v) = rT(u)
\]

So, it is established that \( T \) is a homomorphism.

Conversely, suppose \( T \) is a homomorphism. We will prove that condition (1) holds. From the first, then second property of homomorphism, it follows

\[
T(ru + sv) = T(ru) + T(sv) = rT(u) + sT(v)
\]

So, the equation (1) is established.
Corollary 7.2.3: Linearity, with finite sum

Suppose $V, W$ are two vector spaces and $T : V \rightarrow W$ is a homomorphism (i.e., linear map). Let $u_1, \ldots, u_n \in V$ be $n$ vectors and $c_1, \ldots, c_n \in \mathbb{R}$ be $n$ scalars. Then,

$$T (c_1 u_1 + \cdots + c_n u_n) = c_1 T (u_1) + \cdots + c_n T (u_n).$$

Proof. (We use method of induction, to prove this.) The method has two steps.
(Initialization Step:) When, \( n = 1 \), we need to prove \( T(c_1 u_1) = c_1 T(u_1) \). This follows from the second condition of the definition homomorphisms.

(Induction Step:) We assume the proposition is valid for fewer than \( n - 1 \) summands and prove it for \( n \) summands.

By this assumption

\[
T(c_1 u_1 + \cdots + c_{n-1} u_{n-1}) = c_1 T(u_1) + \cdots + c_{n-1} T(u_{n-1}).
\]

By \( n = 1 \) case: \( T(c_n u_n) = c_n T(u_n) \).
By first condition of the definition homomorphisms

\[ T \left( c_1 u_1 + \cdots + c_{n-1} u_{n-1} + c_n u_n \right) \]

\[ = T \left( c_1 u_1 + \cdots + c_{n-1} u_{n-1} \right) + T \left( c_n u_n \right) \]

\[ = c_1 T \left( u_1 \right) + \cdots + c_{n-1} T \left( u_{n-1} \right) + c_n T \left( u_n \right) \]

The proof is complete.
A basis of a vector space $V$ dictates most of the properties of $V$. The next theorem does exactly the same for Homomorphisms (i.e. linear maps) $T : V \rightarrow W$. 
Theorem 7.2.4: Bases and Linear Maps

Let $V, W$ be two vector spaces. Let $\{v_1, \ldots, v_m\}$ be a basis of $V$ and $w_1, \ldots, w_m \in W$ be $m$ vectors in $W$. Then,

- **(Existence):** Then, there is a homomorphisms $T : V \rightarrow W$ such that

  $$ T(v_1) = w_1, \quad T(v_2) = w_2, \quad \cdots, \quad T(v_m) = w_m. \quad (2) $$

- **(Uniqueness):** The Equation (2) determines a **unique** homomorphisms $T : V \rightarrow W$. 
Proof.

First, we define $T : V \rightarrow W$. Let $x \in V$. By property of bases, there are scalars $c_1, \ldots, c_n \in \mathbb{R}$, such that

$$x = c_1v_1 + c_2v_2 + \cdots + c_nv_m$$

Define $T(x) = c_1w_1 + c_2w_2 + \cdots + c_nw_m$

Since, for a given $x \in V$, $c_1, \ldots, c_m$ are unique (there were no choices involved), the right hand side depends only on $x$. So, $T(x)$ is well defined.

(One needs to justify so called ”well defined-ness”, whenever something is defined.)
To complete the proof of existence, we need to show \( T \) is a homomorphism. Let \( x \in V \) be as above and \( y \in V \) be another vector. There are scalars \( d_1, \ldots, d_n \in \mathbb{R} \), such that

\[
\begin{aligned}
y &= d_1v_1 + \cdots + d_mv_m. \\
T(y) &= d_1w_1 + \cdots + d_mw_m
\end{aligned}
\]

By definition,

\[
\begin{aligned}
x + y &= (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_m + d_m)v_m \\
By Definition, \quad T(x + y) &= (c_1 + d_1)w_1 + \cdots + (c_m + d_m)w_m \\
&= (c_1w_1 + \cdots + c_mw_m) + (d_1w_1 + \cdots + d_mw_m) \\
&= T(x) + T(y).
\end{aligned}
\]

The first (the additive) property of homomorphism is checked.
Now, we prove the $T$ satisfies the second condition:

Let $x \in V$ be as above and $r \in \mathbb{R}$ be a scalars. Then,

$$rx = (rc_1)v_1 + \cdots + (rc_n)v_m$$

By Definition, $T(rx) = (rc_1)w_1 + \cdots + (rc_m)w_m$

$$= (rc_1)w_1 + \cdots + (rc_m)w_m = r(c_1w_1 + \cdots + c_mw_m)$$

$$= rT(x)$$

The second property of homomorphism is checked. So, it is established that $T$ is a homomorphism (i.e. Existence part).
Now we prove uniqueness part. Let $T_1, T_2 : V \longrightarrow W$ be two homomorphism, satisfying Equation (2), meaning

\[
\begin{align*}
T_1 (v_1) &= w_1 & T_1 (v_2) &= w_2, & \cdots, & T_1 (v_m) &= w_m \\
T_2 (v_1) &= w_1 & T_2 (v_2) &= w_2, & \cdots, & T_2 (v_m) &= w_m
\end{align*}
\]

For all $x \in V$, we need to prove $T_1 (x) = T_2 (x)$. As before $x \in V$, we can write

\[\begin{align*}
x &= c_1 v_1 + c_2 v_2 + \cdots + c_m v_m \quad \text{for some } c_1, \ldots, c_m \in \mathbb{R}.
\end{align*}\]
Since $T_1, T_2 : V \rightarrow W$ are homomorphism,

$$T_1 (x) = T_1 (v_1) + c_2 T_1 (v_2) + \cdots + c_m T_1 (v_m)$$

$$= c_1 w_1 + c_2 w_2 + \cdots + c_m w_m$$

Likewise,

$$T_2 (x) = T_2 (v_1) + c_2 T_2 (v_2) + \cdots + c_m T_2 (v_m)$$

$$= c_1 w_1 + c_2 w_2 + \cdots + c_m w_m$$

So, $T_1 (x) = T_2 (x)$. This completes the proof of Part 2 (uniqueness part).
Example 7.2.1

\[ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \quad (3) \]

be the standard basis of \( \mathbb{R}^3 \). Let

\[ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \]

Then, \( v_1, v_2, v_3 \) forms a basis of \( \mathbb{R}^3 \) (we do not prove this)
Let \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) be the homomorphism defined by

\[
T(v_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T(v_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad T(v_3) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},
\]

Compute \( T(e_1), T(e_2), T(e_3) \). More generally, compute

\[
T \begin{pmatrix} a \\ b \\ c \end{pmatrix}.
\]
**Solution:** We will write $e_1, e_2, e_3$, as linear combination of $v_1, v_2, v_3$. First, write $e_1 = \alpha v_1 + \beta v_2 + \gamma v_3$. In matrix form:

$$e_1 = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$
\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
= A^{-1} e_1 = \frac{1}{4} \begin{pmatrix}
2 & 1 & 1 \\
-2 & 1 & 1 \\
0 & -2 & 2
\end{pmatrix} \begin{pmatrix}
1 \\
0
\end{pmatrix} = \begin{pmatrix}
1/2 \\
-1/2 \\
0
\end{pmatrix}
\]

So, 
\[e_1 = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix}
.5 \\
-.5 \\
0
\end{pmatrix} = .5(v_1 - v_2)\]

So 
\[T(e_1) = .5(T(v_1) - T(v_2)) = \begin{pmatrix} 1 \\
0
\end{pmatrix}\]
Now compute $T(e_2)$. As before, write $e_2 = \alpha v_1 + \beta v_2 + \gamma v_3$.

So

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1} e_2 = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} .25 \\ .25 \\ -1 \end{pmatrix}$$

So, $e_2 = (v_1, v_2, v_3) \begin{pmatrix} .25 \\ .25 \\ -1 \end{pmatrix} = .25v_1 + .25v_2 - .5v_3$

So

$$T(e_2) = .25(T(v_1) + .25T(v_2)) - .5T(v_3)$$

$$= .25 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + .25 \begin{pmatrix} -1 \\ 1 \end{pmatrix} - .5 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
Now compute $T(e_3)$. As before, write $e_3 = \alpha v_1 + \beta v_2 + \gamma v_3$

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\end{pmatrix} = A^{-1} e_3 = \frac{1}{4} \begin{pmatrix}
2 & 1 & 1 \\
-2 & 1 & 1 \\
0 & -2 & 2 \\
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
1 \\
\end{pmatrix} = \begin{pmatrix}
.25 \\
.25 \\
.5 \\
\end{pmatrix}
\]

So, $e_3 = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix}
.25 \\
.25 \\
-0.5 \\
\end{pmatrix} = .25v_1 + .25v_2 + 5v_3$

So $T(e_3) = .25(T(v_1) + .25T(v_2)) + .5T(v_3)$

$= .25 \begin{pmatrix} 1 \\
1 \\
\end{pmatrix} + .25 \begin{pmatrix} -1 \\
1 \\
\end{pmatrix} + .5 \begin{pmatrix} 0 \\
-1 \\
\end{pmatrix} = \begin{pmatrix} 0 \\
0 \\
\end{pmatrix}$
Finally,

\[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= ae_1 + be_2 + ce_3.
\]

So,

\[
T\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = aT(e_1) + bT(e_2) + cT(e_3)
\]

\[
= a\begin{pmatrix}
1 \\
0
\end{pmatrix} + b\begin{pmatrix}
0 \\
1
\end{pmatrix} + c\begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{pmatrix}
a \\
b
\end{pmatrix}
\]
Example 7.2.2

Let \( e_1, e_2, e_3 \in \mathbb{R}^3 \) be the standard basis of \( \mathbb{R}^3 \), as in (3) and Let

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3
\end{align*}
\]

Then, \( v_1, v_2, v_3 \) forms a basis of \( \mathbb{R}^3 \) \textit{(we do not prove this)}

Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) be the homomorphism defined by

\[
\begin{align*}
T(v_1) &= \begin{pmatrix} -1 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \\
T(v_2) &= \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \\
T(v_3) &= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}
\end{align*}
\]
Compute $T(e_1), T(e_2), T(e_3)$. More generally, compute $T \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

**Solution:** Steps are very similar to the above problem. We will write $e_1, e_2, e_3$, as linear combination of $v_1, v_2, v_3$. First, write $e_1 = \alpha v_1 + \beta v_2 + \gamma v_3$. 
In matrix form:

\[ e_1 = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \]

\[ A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \]
Continued

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix} = A^{-1} e_1 = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}
\]

So, \(e_1 = (v_1 \ v_2 \ v_3) \begin{pmatrix} .5 \\ -.5 \\ 0 \end{pmatrix} = .5(v_1 - v_2)\)
So \( T(e_1) = \frac{1}{2} (T(v_1) - T(v_2)) \)

\[
\begin{pmatrix}
-1 \\
-1 \\
-1 \\
3
\end{pmatrix}
- \frac{1}{2}
\begin{pmatrix}
1 \\
-1 \\
-1 \\
1
\end{pmatrix}
\begin{pmatrix}
-1 \\
0 \\
0 \\
1
\end{pmatrix}
\]
Now compute $T(e_2)$. As before, write $e_2 = \alpha v_1 + \beta v_2 + \gamma v_3$

\[
\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1} e_2 = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix}
\]

So, $e_2 = (v_1 \hspace{1em} v_2 \hspace{1em} v_3) \begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix} = .25v_1 + .25v_2 - .5v_3$
So \( T(e_2) = .25 \left( T(v_1) + .25 \; T(v_2) \right) - .5 \; T(v_3) \)

\[
= .25 \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \\ -1 & -1 & 0 \end{pmatrix} + .25 \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix} - .5 \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}
\]
Now compute $T(e_3)$. As before, write $e_3 = \alpha v_1 + \beta v_2 + \gamma v_3$

\[
\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1}e_3 = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} .25 \\ .25 \\ .5 \end{pmatrix}
\]

So, $e_3 = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} .25 \\ .25 \\ -.5 \end{pmatrix} = .25v_1 + .25v_2 + 5v_3$
So $T(e_3) = 0.25T(v_1) + 0.25T(v_2) + 0.5T(v_3)$

$$= 0.25 \begin{pmatrix} -1 \\ -1 \\ -1 \\ 3 \end{pmatrix} + 0.25 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + 0.5 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$
Finally, \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ae_1 + be_2 + ce_3 \).

So, \( T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = aT(e_1) + bT(e_2) + cT(e_3) \)

\[
\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \\ a+b+c \end{pmatrix}
\]
Exercises 1

Let \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3 \) be the standard basis of \( \mathbb{R}^3 \), as in (3)

\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} \frac{1}{2} \\ -1 \\ 1 \end{pmatrix}, & \mathbf{v}_2 &= \begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}, & \mathbf{v}_3 &= \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -1 \end{pmatrix} \in \mathbb{R}^3
\end{align*}
\]

Then, \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) form a basis of \( \mathbb{R}^3 \) (need not check). Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be be the homomorphism, defined by

\[
T (\mathbf{v}_1) = \mathbf{v}_1, \quad T (\mathbf{v}_2) = \mathbf{0}, \quad T (\mathbf{v}_3) = \mathbf{0}.
\]

Compute \( T (\mathbf{e}_1), T (\mathbf{e}_2), T (\mathbf{e}_3) \), and in particular \( T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \).
Exercise 2

Let $\mathbb{P}_2(\mathbb{R})$ be the vector space of all polynomials $f$ with $\deg(f) \leq 2$. Then,

$p_1(x) = 1 + x + x^2$, $p_2(x) = x + x^2$, $p_3(x) = x^2$ is a basis of $\mathbb{P}_2(\mathbb{R})$.

Define the homomorphism $T : \mathbb{P}_2(\mathbb{R}) \rightarrow \mathbb{M}_2(\mathbb{R})$ by

$$T(p_1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \ T(p_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ T(p_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Compute $T(1)$, $T(x)$, $T(x^2)$ and in general $T(a + bx + cx^2)$. 
Recall, given a matrix $A \in \mathbb{M}_{m \times n}(\mathbb{R})$, there is a homomorphism

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ defined by } T(x) = Ax \quad \forall \ x \in \mathbb{R}^n.$$

We demonstrate that, any homomorphism $T : V \longrightarrow W$ of vectors spaces, with finite dimension, are determined by matrices.
Theorem 7.2.5: Matrices to Homomorphisms

Let $V, W$ be two vector spaces.

- Let $v_1, v_2, \ldots, v_m \in V$ be a basis of $V$ and $w_1, w_2, \ldots, w_n \in W$ be elements in $W$.

- Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{M}_{m \times n}(\mathbb{R})$$
Then, there is a homomorphism $T_A : V \longrightarrow W$ such that

$$
\begin{align*}
T_A(v_1) &= a_{11}w_1 + a_{12}w_2 + \cdots + a_{1n}w_n \\
T_A(v_2) &= a_{21}w_1 + a_{22}w_2 + \cdots + a_{2n}w_n \\
& \quad \cdots \\
T_A(v_m) &= a_{m1}w_1 + a_{m2}w_2 + \cdots + a_{mn}w_n
\end{align*}
$$

(4)

In matrix notation,

$$
\begin{pmatrix}
T_A(v_1) \\
T_A(v_2) \\
\vdots \\
T_A(v_m)
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{pmatrix}
$$

(5)
Proof. It follows from Theorem 7.2.4 and equation (4) and theorem above.
We Remark:

- The notation $T_A$ was chosen, with subscript $A$, to show its dependence on $A$, and for future reference.

- $T_A$, also, depend on the basis $\{v_1, \ldots, v_m \in V\}$ and elements $\{w_1, \ldots, w_m \in W\}$. That means, if we change the basis $v_i$, or elements $\{w_j\}$ the homomorphism $T_A$ we get will be different.
Suppose $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$. Let $\{v_1, \ldots, v_m \in V\}$ be the standard basis of $V = \mathbb{R}^m$ and $\{w_1, \ldots, w_m \in W\}$ be the standard basis of $W = \mathbb{R}^n$ (as in Equation 3). Then,

$$T_A(x) = A^t x \quad \forall \ x \in \mathbb{R}^m.$$ 

This example was discussed before.

A converse of the above is also valid as follows.
Theorem 7.2.6: Homomorphisms to Matrices

Let \( V, W \) be two vector spaces. Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V \) be a basis of \( V \) and \( \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \in W \) be basis in \( W \).

Let \( T : V \rightarrow W \) be a homomorphism. Since, \( \{ \mathbf{w}_j \} \) is a basis of \( W \), we can back track the above steps and write uniquely:

\[
\begin{align*}
T(\mathbf{v}_1) &= a_{11}\mathbf{w}_1 + a_{12}\mathbf{w}_2 + \cdots + a_{1n}\mathbf{w}_n \\
T(\mathbf{v}_2) &= a_{21}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{2n}\mathbf{w}_n \\
&\quad \ldots \\
T(\mathbf{v}_m) &= a_{m1}\mathbf{w}_1 + a_{m2}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_n
\end{align*}
\]

(6)

with unique \( a_{ij} \in \mathbb{R} \).
This way we get a well defined matrix

\[
A_T = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\] (7)

We remark, the $A_T$ depends on the choice of bases of $V$ and of $W$, as above.
Theorem 7.2.7: The Correspondence

Let $V, W$ be two vector spaces. Let $v_1, v_2, \ldots, v_m$ be a basis of $V$ and $w_1, w_2, \ldots, w_n$ be basis in $W$. Let $\mathcal{L}(V, W)$ be the set of all homomorphisms $V \rightarrow W$.

- Define $\varphi : \mathcal{L}(V, W) \rightarrow M_{m \times n}(\mathbb{R})$ by $\varphi(T) = A_T$, where $A_T \in M_{m \times n}(\mathbb{R})$ is the matrix as in Theorem 7.2.6. Then, $\varphi$ is a well-defined bijective correspondence.

- Define $\psi : M_{m \times n}(\mathbb{R}) \rightarrow \mathcal{L}(V, W)$ by $\psi(A) = T_A$, where $T_A \in \mathcal{L}(V, W)$ is as in Theorem 7.2.5. Then, $\psi$ and $\varphi$ are the inverses of each other.
Proof. It follows from the above discussions on the definitions of $A_T$ and $T_A$. We skip the details of the proof.

Remark. We comment that $A_T$ and $T_A$ depend on the choices of bases $\{v_i\}$ of $V$ and $\{w_j\}$ of $W$. Hence $\varphi$ and $\psi$ would also do the same.
Definitions and Theorem 7.2.8

Definitions. Let $V, W$ be two vector spaces and $T : V \rightarrow W$ is a homomorphism. Then, define

\[
\mathcal{N}(T) = \{ v \in V : T(v) = 0_W \}.
\]

\[
\mathcal{R}(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \}.
\]

Then, $\mathcal{N}(T)$ is a subspace of $V$. This subspace $\mathcal{N}(T)$ is called the Null Space of $T$.

Then, $\mathcal{R}(T)$ is a subspace of $W$. This subspace $\mathcal{R}(T)$ is called the Range of $T$.

Proof. Skip
As before $T : V \rightarrow W$ be a homomorphism. Also, define

- $\text{Nullity}(T) = \dim \mathcal{N}(T)$.
- $\text{rank}(T) = \dim \mathcal{R}(T)$.
Motivating Example

Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ and $T = T_A : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be as above.

- Then, $\mathcal{N}(T) = \mathcal{N}(A)$. In words, the Null space of $T$ and Null space of $A$ are same.
  Therefore, $\text{Nullity}(T) = \text{Nullity}(A^t)$.

- Also, the range $\mathcal{R}(T)$ is equal to the column space of $A$.
  Therefore, $\text{rank}(T) = \text{rank}(A)$. 
Theorem 7.2.9: Injective Homomorphisms

Let \( V, W \) be vector spaces and \( T : V \rightarrow W \) be a homomorphism. Then, \( T \) is injective if and only if the null space \( \mathcal{N}(T) = \{0\} \).

**Proof.** (\( \Rightarrow \)): Suppose \( T \) is injective and \( x \in \mathcal{N}(T) \). So, \( T(x) = 0_W = T(0) \). By injectivity of \( T \), \( x = 0 \). So, \( \mathcal{N}(T) \subseteq \{0\} \). So, \( \mathcal{N}(T) = \{0\} \)

(\( \Leftarrow \)): Suppose \( \mathcal{N}(T) = \{0\} \). Let \( x_1, x_2 \in V \) and \( T(x_1) = T(x_2) \). Then, \( T (x_1 - x_2) = 0_W \). So, \( x_1 - x_2 \in \mathcal{N}(T) = \{0\} \). So, \( x_1 - x_2 = 0 \) and \( x_1 = x_2 \). So, \( T \) is injective.
Theorem 7.2.10: Bijective Homomorphisms

Let $V, W$ be a vector spaces and $T : V \rightarrow W$ be homomorphism. Then, the following three statements are equivalent.

1. $T$ is bijective.
2. The null space $\mathcal{N}(T) = \{0\}$ and range $\mathcal{R}(T) = W$.
3. $Nullity(T) = \{0\}$ and range $\mathcal{R}(T) = W$.

Proof. Follows from the above.
Isomorphisms

**Definition.** Let $V, W$ be a vector spaces. A bijective homomorphism $T : V \to W$ is also called **isomorphism**. When there is such an isomorphism, we say $V$ and $W$ are **isomorphic**.

**Theorem.** Let $V, W$ be a vector spaces and $T : V \to W$ is an **isomorphism**. Let $G : W \to V$ be the set theoretic inverse of $T$. Then, $G$ is also an isomorphism.

**Proof.** Skip.
If $V$ and $W$, are isomorphic, then properties (Vector-space related) of $V$ translates to properties of $W$, and conversely. So, they can be treated as "same". For Example:

Suppose $T : V \rightarrow W$ is an isomorphism.

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of $V$. Then, $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is a basis of $W$.

- So, $\dim V = \dim W$. 
Let $V$ be a vector space and $W \subseteq V$ be a subspace of $V$. Assume $\dim V < \infty$. Then, $W = V$ if and only if $\dim W = \dim V$. In particular, if $V$ and $W$ are isomorphic, then $\dim V = \dim W$.

**Proof.** It is obvious, if $W = V$ then, $\dim W = \dim V$. Now, assume $\dim W = \dim V = m$. Let $w_1, \ldots, w_m$ be a basis of $W$. If $W \neq V$, there is $v \in V$ and $v \notin W$. Then, $w_1, \ldots, w_m, v$ are linearly independent. This contradicts that $\dim V = m$. So, $V = W$. 

\[ \Box \]
Theorem 7.2.12: Isomorphism and Dimension

Let $V, W$ be vector spaces, with $\dim V < \infty$, $\dim W < \infty$. Then, $V$ and $W$ are isomorphic if and only if $\dim V = \dim W$.

**Proof.** As was established above, if $V, W$ are isomorphic then $\dim V = \dim W$. Now, suppose $\dim V = \dim W = n$.

Let $v_1, \ldots, v_n$ is a basis of $V$, and $w_1, \ldots, w_n$ is a basis of $W$.

Let $T : V \to W$ be the homomorphism, such that

$$T(v_1) = w_1, \ldots, T(v_n) = w_n$$

It is easy to see $T$ is an isomorphism.
Suppose $V$ is a vector space with $\dim V = n$. Let $v_1, \ldots, v_n$ is a basis of $V$. Let $e_1, \ldots, e_n$ be the standard basis basis of $\mathbb{R}^n$. Then, the homomorphism $f : \mathbb{R}^n \rightarrow V$, determined by,

$$f(e_1) = v_1, \ldots, f(e_n) = v_n$$

is an isomorphism. We would call this isomorphism the standard isomorphism.
Theorem 7.2.14: Nullity-Rank Theorem

**Theorem.** Let $V, W$ be vector spaces, with $\dim V = m < \infty$, $\dim W = n < \infty$. Let $T : V \rightarrow W$ be a homomorphism. Then,

$$\text{Nullity}(T) + \text{rank}(T) = \dim V = m.$$  

**Proof.** Fix a basis $v_1, \ldots, v_m$ of $V$ and a basis $w_1, \ldots, w_n$ of $W$. Let $A := A_T \in \mathbb{M}_{m \times n}(\mathbb{R})$ be the matrix of $T$, with respect to these bases. Let $A^t \in \mathbb{M}_{n \times m}(\mathbb{R})$ denote the transpose of $A$. 

Close inspection shows, the diagram

\[
\begin{array}{c}
\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n \\
\longrightarrow \\
\downarrow f \quad \downarrow g \\
V \xrightarrow{T} W
\end{array}
\]

where \( f, g \) are the standard isomorphisms. The restrictions of \( f \) establishes an isomorphism \( f_0 : \mathcal{N}(T_A) \to \mathcal{N}(T) \). So,

\[
\text{Nullity}(A^t) = \dim \mathcal{N}(T_A) = \dim \mathcal{N}(T) = \text{Nullity}(T).
\]
Likewise, restrictions of $g$ establishes an isomorphism
$g_0 : \mathcal{R}(T_{A^t}) \longrightarrow \mathcal{R}(T)$. So,

$$rank(A^t) = \dim \mathcal{R}(T_{A^t}) = \dim \mathcal{R}(T) = rank(T).$$

Recall, we proved

$$Nullity(A^t) + rank(A^t) = m \quad \text{(no of columns of) } A^t$$

So,

$$Nullity(T) + rank(T) = \dim V = m.$$
We worked with eigenvalues and eigenvectors for matrices. Now, vector spaces $V$ and linear transformations $T : V \rightarrow V$, we define eigenvalues and eigenvectors.

Let $V$ be a vector space and $T : V \rightarrow V$ be a Linear Transformation. A scalar $\lambda \in \mathbb{R}$ is said to be a eigenvalue of $T$, if $T(x) = \lambda x$ for some $x \in V$, with $x \neq 0$. In this case, $x$ would be called eigenvector, of $T$, corresponding to $\lambda$. 
Theorem 7.2.15: Eigen Value in Two Ways

Let $V$ be a vector space and $T : V \rightarrow V$ be a Linear Transformation. Assume $\dim V = n < \infty$. Let $B = \{v_1, \ldots, v_n\}$ be a basis of $V$. Let $A \in \mathbb{M}_{n \times n}(\mathbb{R})$ be the matrix of $T$, with respect to the basis $B$, on two sides. That means,

$$
\begin{pmatrix}
T(v_1) \\
T(v_2) \\
\vdots \\
T(v_n)
\end{pmatrix} = A
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix}
$$
Also, let $f : \mathbb{R}^n \rightarrow V$ be the standard isomorphism, and $g : V \rightarrow \mathbb{R}^n$ be the inverse of $f$. So, $f, g$ are determined by

$$
\begin{align*}
    f(e_1) &= v_1, \ldots, f(e_n) = v_n \\
g(v_1) &= e_1, \ldots, g(v_n) = e_n
\end{align*}
$$

Then, for $\lambda \in \mathbb{R}$, the following three conditions are equivalent:

- $\lambda \in \mathbb{R}$ is an eigen value of $T$
- $\lambda \in \mathbb{R}$ is an eigen value of $A$.
- $\lambda \in \mathbb{R}$ is an eigen value of $A^t$. 
Further, corresponding an eigenvalue $\lambda$ of $T$, $x \in \mathbb{R}^n$ is an eigenvector of $A^t$ if and only if $f(x)$ is an eigenvector of $T$.

**Proof.** Proof follows from the following commutative diagram:

![Commutative Diagram]

More explicitly,

$$A^t x = \lambda x \iff f(A^t x) = f(\lambda x) \iff T(f(x)) = \lambda f(x).$$

The proof is complete. $\blacksquare$
Theorem 7.2.16: Change of Basis

Let $V$ be a vector space, with $\dim V = n$ and $T : V \rightarrow V$ be a linear transformation. Let $B_1 = \{v_1, \ldots, v_n\}$, $B_2 = \{w_1, \ldots, w_n\}$ two bases of $V$.

Since both $B_1$ and $B_2$ are both bases, there is an invertible matrix $P$, expressing $B_1$ in terms of $B_2$, as follows:

$$
\begin{pmatrix}
    v_1 \\
    v_2 \\
    \vdots \\
    v_n
\end{pmatrix} = P
\begin{pmatrix}
    w_1 \\
    w_2 \\
    \vdots \\
    w_n
\end{pmatrix}
$$

(8)
Using the basis $\mathcal{B}_1$ (respectively, $\mathcal{B}_2$), for both domain and codomain, we have $A_T, B_T$, as follows:

\[
\begin{pmatrix}
T(v_1) \\
T(v_2) \\
\vdots \\
T(v_n)
\end{pmatrix}
= A_T
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix},
\begin{pmatrix}
T(w_1) \\
T(w_2) \\
\vdots \\
T(w_n)
\end{pmatrix}
= B_T
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{pmatrix}
\]

(9)
These three matrices are related as follows:

\[ B_T = P^{-1} A_T P \]

This is called the Change of Basis Formula.

**Proof.** Rewrite the first equation (9):

\[
T \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = A_T \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}
\]
Continued

Using the equation (8), in this,

\[ T \left( P \begin{pmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{pmatrix} \right) = A_T P \begin{pmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{pmatrix} \]

So,

\[ PT \begin{pmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{pmatrix} = A_T P \begin{pmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{pmatrix} \]
Comparing the second equation in (9), we have

\[ B_T = P^{-1} A_T P. \]

This establishes the Change of Basis Formula.