Math 790 **Test 8 (Takehome)** Satya Mandal

Fall 05 Each Problem 10 points Due on: November 16, 2005

Unless otherwise stated, $\mathbb{F}$ is a field and V will denote a vector space over $\mathbb{F}$ with finite dimension $\dim V = n$. .

1. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V,V)$ is an operator on V. Suppose W_0 is an $T-$invariant subspace of V and $v \in V \setminus W_0$. Write $W_1 = W_0 + \mathbb{F}[T]v$ and

$$I = \{f \in \mathbb{F}[X] : f(T)v \in W_0\}.$$

Prove that

   (a) I is a proper ideal,

   (b) Let f be the MMP of I. Prove that $\dim W_1 = \dim W_0 + degree(f)$.

2. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V,V)$ is an operator on V. Prove that V is cyclic if and only if V has a basis E such that with respect to E the matrix of T is the companion matrix of a monic polynomial p.

3. Let p be a non-constant monic polynomial and A be the companion matrix of p. Prove that p is both the MMP of A and the characteristic polynomial of A.

4. Let $T : \mathbb{F}^3 \to \mathbb{F}^3$ be th operator defined by

$$T(X) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} X$$

Prove that $\mathbb{F}^3$ is not $T-$cyclic.

5. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V,V)$ is an operator on V. Suppose $w_1, \ldots, w_r$ be such that

   (a) $V = \mathbb{F}[T]w_1 \oplus \mathbb{F}[T]w_2 \oplus \cdots \oplus \mathbb{F}[T]w_r$.

   (b) Let p_i be the MMP of w_i. Assume that $p_k \mid p_{k-1}$ for $k = 2, \ldots, r$.

Prove $ann(T) = ann(w_1) = \mathbb{F}[X]p_1$.

6. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V, V)$ is an operator on V. Assume W_0 is a $T-$admissible set and $V = W_0 + \mathbb{F}[T]v$ for some $v \in V$. Find $w \in V$ such that
   (a) $V = W_0 \oplus \mathbb{F}[T]w$.
   (b) Let p be the MMP of w. Prove that p is unique. That means, if $V = W_0 \oplus \mathbb{F}[T]w'$ for some w' then MMP of w' is p.

7. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V, V)$ is an operator on V. Then V is $T-$cyclic if and only if the characteristic polynomial and the MMP of T are identical.

8. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V, V)$ is an operator on V. Let P be the MMP of T and Q be the characteristic polynomial of T. Let p be an irreducible polynomial. Prove that
$$p \mid P \Longleftrightarrow p \mid Q.$$

9. Given an example (with justification) of a $2 \times 2-$matrix A whose characteristic polynomial is $(1 - X)^2$.

10. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V, V)$ is an operator on V. Suppose the characteristic polynomial Q of T factors completely into linear factors:
$$Q = (X - c_1)^{d_1}(X - c_2)^{d_2}(X - c_k)^{d_k}$$
where $c_1, \ldots, c_k$ are the distinct eigen values of T. Describe the Jordan matrix of T and give a short outline of the proof of existance.