Math 790	**Test 9 (Takehome)**	Satya Mandal
Fall 05	Each part of each Problem is worth 5 points	Due on: December 8, 2005

Unless otherwise stated, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and V will denote an inner product space over $\mathbb{F}$.

1. Let V be an inner product space over $\mathbb{F}$..

   (a) Prove the parallelogram law that

   $$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2 .$$

   (b) Prove the Cauchy-Schwartz inequality

   $$| (x, y) | \quad \leq \quad \| x \| \| y \|$$

   and that the equality holds if an only if

   $$y = \frac{(y, x)}{\| x \|^2} x.$$

2. Consider $V = \mathbb{C}^n$, with the usual inner product. Think of the elements of V as row vectors.

   Let $v_1, v_2, \ldots, v_n \in V$ and let

   $$A = \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{pmatrix}$$

   (a) Prove that $v_1, v_2, \ldots, v_n$ forms a basis if and only if A invertible.

   (b) Prove that $v_1, v_2, \ldots, v_n$ forms an orthonormal basis if and only if A is an unitary matrix (i.e. $A^* A = I$).

   (c) Formulate the similar statements about vectors in $\mathbb{R}^n$ and orthogonal matrices.

3. Let V be an inner product space over $\mathbb{F}$. Suppose $E = \{e_1, \ldots, e_n\}$ is an orthonormal basis.

(a) Suppose $v \in V$ and $v = a_1 e_1 + \cdots + a_n e_n$ where $c_i \in \mathbb{F}$. Compute (with proof) c_i in terms of E and v.

(b) Suppose $\mathcal{E} = \{\epsilon_1, \ldots, \epsilon_n\}$ be another orthonormal basis. Let

$$(e_1, \ldots, e_n) = (\epsilon_1, \ldots, \epsilon_n)P$$

for some amtrix $P \in \mathbb{M}_n(\mathbb{F})$. Prove that P is a unitary matrix.

4. Let V be an inner product space over $\mathbb{F}$. Suppose W is a subspace of V and $E = \{e_1, \ldots, e_n\}$ is an orthonormal basis of W. For $v \in V$ define

$$\pi(v) = \sum_{k=1}^{n} (v, e_k) e_k.$$

Prove (do not quote or use any theorem) the following:

(a) $(v - \pi(v)) \perp w$ for all $w \in W$,

(b) and

$$\| (v - \pi(v)) \| \quad \leq \quad \| (v - w) \| \quad for \quad all \quad w \in W.$$

(c) Also prove that π is a projection and $V = W \oplus W^{\perp}$.

5. Let V be an inner product space over $\mathbb{F}$ with $dim V = n$ finite. Let $T \in L(V, V)$ be an operator.

(a) Prove that there is unique operator $T^* \in L(V, V)$ such that

$$(T(x), y) = (x, T^* y)) \quad for \quad all \quad x, y \in V.$$

(b) $E = \{e_1, \ldots, e_n\}$ is an orthonormal basis of V. Let A matrix of T with respect to E. Prove that matrix of T^* with respect to E is A^*.

(c) Give an example of an innner product space V, a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$, and an operator $T \in L(V, V)$ so that above (5a) fails OR that matrix, with respect to $\mathcal{B}$, of $T^* \neq A^*$ where A is the matrix of T with respect to $\mathcal{B}$.

6. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let V be an finite dimensional inner product space over $\mathbb{F}$. Let $T, U \in L(V, V)$ be two linear operator and $c \in \mathbb{F}$. Then

   (a) $(T + U)^* = T^* + U^*$,

   (b) $(cT)^* = \overline{c}T^*$,

   (c) $(TU)^* = U^*T^*$,

   (d) $(T^*)^* = T$.

7. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let V be an inner product space over $\mathbb{F}$. Let $T, U \in L(V, V)$ be two linear operator.

   (a) Prove that T preserves inner product if and only if T preserves norm.

   (b) U is unitary if and only if the adjoint U^* of U exists and $UU^* = U^*U = I$.

8. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let V be a finite dimensional (with $\dim V = n$) inner product space over $\mathbb{F}$.

   (a) Let $T \in L(V, V)$ be self-adjoint operator. Prove that V has an orthonormal basis $E = \{e_1, \ldots, e_n\}$ such that each e_i is an eigen vector of T.

   (b) Let $A \in \mathbb{M}_n(\mathbb{F})$ be an Hermitian (self-adjoint) matrix. Prove that there is an unitary matrix $P \in \mathbb{M}_n(\mathbb{F})$ such that $P^{-1}AP$ is a diagonal matrix.

9. Let V be a finite dimensional inner product space over $\mathbb{C}$.

   (a) Let $T \in L(V, V)$ be a normal operator on V. Then there is an orthonormal basis $E = \{e_1, \ldots, e_n\}$ such that each e_i is an eigen vector of T.

   (b) Let $A \in \mathbb{M}_n(\mathbb{C})$ be a NORMAL matrix. Then there is an unitary matrix $U \in \mathcal{U}(n) \subseteq \mathbb{M}_n(\mathbb{C})$, such that $U^{-1}AU$ is a diagonal matrix.