Annihilators : Linear Algebra Notes

Satya Mandal

September 21, 2005

Let F be a filed and V be vector space over F with $\dim(V) = n < \infty$. As usual V^* will denote the dual space of V and $V^{**} = (V^*)^*$.

1. **Definition 0.1** *For a subset $S \subseteq V$ define the annihilator*

$$ann(S) = \{f \in V^* : f(v) = 0 \ for \ allv \in S\}.$$

We also use the notation $S^0 = ann(S)$.

We may also, temporarily, use the notation

$$ann(V^* : S) = ann(S)$$

to underscore the fact that $ann(V^* : S)$ is a subspece of V^*.

2. First note that for $S \subseteq V$, the annihilator $ann(S) \subseteq V^*$ is a subspace of the dual space.

3. Also note for $S \subseteq V$, if $W = Span(S)$ then

$$ann(S) = ann(W).$$

4. Now let $\mathcal{S} \subseteq V^*$. Then according to above definition, annihilator of $\mathcal{S}$ is $ann(S) = ann(V^{**} : \mathcal{S})$ is a subspace of V^{**}.

In this case, there is another natural annihilator of $\mathcal{S}$ as a subspace of V as follows:

$$ann(V : \mathcal{S}) = \{v \in V : f(v) = 0 \ for \ all \ f \in \mathcal{S}\}.$$

It is possible to mix up two annihilator of $\mathcal{S}$. The first one is a subspace of the double dual V^{**} and the second one is the subspace of V.

We will justify that these two annihilators are same via the natural identification of V and V^{**}.

5. Let $L : V \xrightarrow{\sim} V^{**}$ be the natural isomorphism.

6. Let $\mathcal{S} \subseteq V^*$. Then

$$L(ann(V : \mathcal{S})) = ann(V^{**} : \mathcal{S}).$$

Proof. Let $v \in ann(V : \mathcal{S})$. Then $f(v) = 0$ for all $f \in \mathcal{S}$.

Hence $L(v)(f) = f(v) = 0$ for all $f \in \mathcal{S}$.

So, $L(v) \in ann(V^{**} : \mathcal{S})$. Therefore

$$L(ann(V : \mathcal{S})) \subseteq ann(V^{**} : \mathcal{S}).$$

Now let $G \in ann(V^{**} : \mathcal{S})$. Since L is an isomorphism, we have $G = L(v)$ for some $v \in V$. We need to show that $v \in ann(V : \mathcal{S})$. But for $f \in \mathcal{S}$ we have $0 = G(f) = L(v)(f) = f(v)$. Therefore $v \in ann(V : \mathcal{S})$. So,

$$ann(V^{**} : \mathcal{S}) \subseteq L(ann(V : \mathcal{S})).$$

Hence the proof is complete.

7. Now, for a subspace $W \subseteq V$, the annihilator $W^0 = ann(V : W)$ has two annihilators. They are $ann(V : W^0)$ and $ann(V^{**} : W^0)$.

It follows from above that

$$L(ann(V : W^0)) = ann(V^{**} : W^0).$$

Since, L is a linear isomorphism, we also have

$$\dim(ann(V : W^0)) = \dim(ann(V^{**} : W^0)).$$

8. **Theorem 0.1** *For a subspace $W \subseteq V$, we have*

$$W = ann(V : ann(V, W))$$

also written as

$$W = W^{00}.$$

Proof. Write $U = ann(V : ann(V : W))$. It is easy to see that $W \subseteq U$. Therefore, it is enough to show that $\dim(W) = \dim(U)$. We have

$$\dim(W) + \dim(ann(V : W)) = \dim(V).$$

Also, by the same theorem,

$$\dim(ann(V : W)) + \dim(ann(V^{**} : ann(V, W)) = \dim(V^{**}) = \dim(V).$$

It follows that

$$\dim(W) = \dim(ann(V^{**} : ann(V, W)) = \dim(ann(V : ann(V : W)) = \dim(U).$$

So the proof is complete.

Lemma 0.1 *Suppose V is vector space of finite dimension,* $\dim V = n$*, over $\mathbb{F}$. Let $f, g \in V^*$ be two linear functionals. let N_f be the null space of f and N_g be the null space of g.*

Then, $N_f \subseteq N_g$ if and only if $g = cf$ for some $c \in \mathbb{F}$.

Proof. ($\Leftarrow$:) Obvious.
($\Rightarrow$:) If $g = 0$ then $g = cf$ with $c = 0$. So, we assume that $g \neq 0$. So, $\dim(N_g) = n-1$. Since $N_f \subseteq N_g$, we have $f \neq 0$ and $\dim(N_f) = n-1$. Therefore, $N_f = N_g = N(say)$.

Now, pick $e \notin N$. Since $\dim(V) = n$, it follows that $V = N + \mathbb{F}e$. Also, $f(e) \neq 0$ and $g(e) \neq 0$. Write $c = g(e)/f(e)$. Claim that

$$g = cf.$$

First note, $g(e) = cf(e)$. Now, for $x \in V$, we have $x = y + \lambda e$ for some $y \in N$ and $\lambda \in \mathbb{F}$. Therefore

$$g(x) = g(y) + \lambda g(e) = \lambda g(e) = \lambda c f(e) = c(f(y + \lambda e)) = cf(x).$$

So, the proof is complete.

Following is Theorem 20, page 110.

Theorem 0.2 *Suppose V is vector space of finite dimension,* $\dim V = n$, *over $\mathbb{F}$. Let $g, f_1, \ldots, f_r \in V^*$ be linear functionals. Let N be the null space of g and N_i be the null space of f_i.*

Then, $N_1 \cap N_2 \cap \cdots \cap N_r \subseteq N$ if and only if $g = \sum_{i=1}^{r} c_i f_i$ for some $c_i \in \mathbb{F}$.

Proof. ($\Leftarrow$:) Obvious.
($\Rightarrow$:) We use induction on r to prove this part. Case $r = 1$, is the above Lemma 0.1.

Now, we assume the validity of the thoerem for $r - 1$ functionals and prove it for r. Write $V' = N_r$.

Let

$$g' = G_{|V'}, f_1' = (f_1)_{|V'}, \ldots, f_{r-1}' = (f_{r-1})_{|V'},$$

be the restrictions of the respective functionals to V'. Induction applies for these functionals and it follows that

$$g' = \sum_{i=1}^{r-1} c_i f_i'$$

for some $c_i \in \mathbb{F}$. This means, for all $x \in V' = N_r$, we have

$$g(x) = \sum_{i=1}^{r-1} c_i f_i(x).$$

This means, for all $x \in V' = N_r$, we have

$$g(x) = \sum_{i=1}^{r-1} c_i f_i(x).$$

Write

$$h = g - \sum_{i=1}^{r-1} c_i f_i.$$

Then $N_r = V' \subseteq Null - Space(h)$. By the case $r = 1$ (or by Lemma 0.1) It follows that $h = c_r f_r$ for some $c_r \in \mathbb{F}$. Hence

$$g = \sum_{i=1}^{r-1} c_i f_i + h = \sum_{i=1}^{r} c_i f_i$$

and the proof is complete.