Math 790 **Test 6 (Takehome)** Satya Mandal

Fall 05 Each Problem 10 points Due on: October 24, 2005

Unless otherwise stated, $\mathbb{F}$ is a field and V will denote a vector space over $\mathbb{F}$ with finite dimension $\dim V = n$. .

1. Suppose $T \in L(V, V)$ is a linear operator and $c \in \mathbb{F}$ be a scalar. Prove that c is an eigen value of T if and only if $\det(T - cI) = 0$.

2. Suppose $T \in L(V, V)$ is a linear operator. Suppose $c_1, c_2, \ldots, c_k \in \mathbb{F}$ be distinct eigen values of T. Let $N(c_i) = \{v \in V : T(v) = c_i v\}$ be the eigen space of c_i. Let $W = N(c_1) + \cdots + N(c_{k-1})$ Prove that $W \cap N(c_k) = \{0\}$.

3. Suppose V_1, V_2 be two subspaces of V. Prove that $V = V_1 \oplus V_2$ if and only if $V = V_1 + V_2$ and $\dim V_1 + \dim V_2 = \dim V$.

4. Suppose $T \in L(V, V)$ is a linear operator and $c_1, c_2, \ldots, c_k \in \mathbb{F}$ are the distinct eigen values of T. For $i = 1, \ldots, k$, let $W_i = N(c_i)$ be the eigen space of c_i.

   Prove that T is diagonalizable if and only if $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$.

5. Suppose $T \in L(V, V)$ is a linear operator and $c_1, c_2, \ldots, c_k \in \mathbb{F}$ are the distinct eigen values of T. For $i = 1, \ldots, k$, let $W_i = N(c_i)$ be the eigen space of c_i and $d_i = \dim(W_i)$.

   Prove $\dim(V) = d_1 + d_2 + \cdots + d_k$ if and only if the characteristic polynomial f of T is given by

   $$f = (X - c_1)^{d_1}(X - c_2)^{d_2} \cdots (X - c_k)^{d_k}.$$

6. Suppose $T \in L(V, V)$ is a linear operator. Let $p(X)$ be the minimal polynomial of T and $g(X)$ be the characteristic polynomial of T. For a scalar $c \in \mathbb{F}$, prove that $p(c) = 0$ if and only if $g(c) = 0$.

7. Suppose $T \in L(V, V)$ is a linear operator. Let $p(X)$ be the minimal polynomial of T and $g(X)$ be the characteristic polynomial of T. Prove that $g(T) = 0$. Also prove that $p \mid g$.

8. Let A be an $n \times n$ matrix with entries in $\mathbb{F}$. Ler $T : \mathbb{F}^n \to \mathbb{F}^n$ be a operator defined by $T(X) = AX$. Prove that T is diagonalizable if an only if $PAP^{-1} = \Delta$, where Δ is a diagonal matrix and $P \in GL_n(\mathbb{F})$.

9. Let A be an $n \times n$ matrix with entries in $\mathbb{F}$. Suppose the characteristic polynomial of A is given by $f = (X - c_1)(X - c_2)\cdots(X - c_n)$. Prove that $trace(A) = c_1 + c_2 + \cdots + c_n$.

Suggestion: I am not assigning any numerical problem, regarding which matrix is diagonalizable and which is not. Read all the examples from the textbook.