Math 790	**Test 7 (Takehome)**	Satya Mandal
Fall 05	Each Problem 10 points	Due on: October 31, 2005

Unless otherwise stated, $\mathbb{F}$ is a field and V will denote a vector space over $\mathbb{F}$ with finite dimension $\dim V = n$. .

1. Suppose V is vector space over $\mathbb{F}$ with finite $\dim(V) = n$. Let $T \in L(V, V)$ be a linear operator. Suppose W is a $T-$invariant subspace of W and $T' = T_{|W}$ be the restriction.

   (a) Let q be the characteristic polynomial of T and Q be the characteristic polynomial of T'. Prove that $Q \mid q$.

   (b) Likewise, let p bet the MMP of T and P be the MMP of T'. Prove that $P \mid p$.

2. Suppose V is vector space over $\mathbb{F}$ with finite $\dim(V) = n$. Let $T \in L(V, V)$ be a linear operator. Suppose there is a basis E of V such that the matrix A of T with respect to E is upper triangular. Prove that there is another basis $\mathcal{E}$, such that with respect to $\mathcal{E}$, the matrix of T is lower triangular.

3. Let V be a vector space over $\mathbb{F}$ with with finite dimension $\dim V = 3$ and $T : V \to V$ be a linear operator on V. Prove that T is triangulable if and only if the minimal polynomial p of T is a product of linear factors.

4. Let V be a finite dimensional vector space over a field $\mathbb{F}$.

   (a) Let $e_1, e_2, \ldots, e_k$ be elements of V. Prove that $e_1, e_2, \ldots, e_k$ are linearly independent if and only if $j = 1, \ldots, k$, we have $e_j \notin Span(e_1, e_2, \ldots, e_{j-1})$.

   (b) Let $W_1, \ldots, W_k$ be subspaces of V. Prove that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ if and only if $V = W_1 + W_2 + \cdots + W_k$ and for each $j = 2, \ldots, k$, we have
   $$(W_1 + \cdots + W_{j-1}) \cap W_j = \{0\}.$$

5. Let V be a finite dimensional vector space over a field $\mathbb{F}$. Suppose $E_1, \ldots, E_k$ are k linear operators on V satisfying all the conditions:

(a) $E_iE_j = 0 \quad \forall \quad i \neq j$.

(b) $E_1 + E_2 + \cdots + E_k = I$.

Write $W_i = E_i(V)$. Prove that E_i is a projection and

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

6. Let V be a finite dimensional vector space over a field $\mathbb{F}$ and W be a subspace of V. Prove that there is subspace U of V such that $V = W \oplus U$.

7. Let $V = \mathbb{R}^2$.

   (a) Write down the the projection $\pi : V \to V$ to the line $y = x$.

   (b) Let $= (0, 1) \in V$. Write down the projection $p : V \setminus \{P\} \to V$ to the $x-$axis from the point P. Is it linear?

8. Let V be a vector space over $\mathbb{F}$ with finite dimension $\dim V = n$ and $T : V \to V$ be a linear operator on V. Let p be the minimal monic polynomial (MMP) of T and

$$p = p_1^{r_1} p_1^{r_2} \cdots p_1^{r_k}$$

where $r_i > 0$ and p_i are distinct irreducible monic polynomials in $\mathbb{F}[X]$. Let

$$W_i = \{v \in V : p_i(T)^{r_i}(v) = 0\}$$

be the null space of $p_i(T)^{r_i}$.

   (a) Prove that W_1 is invariant under T.

   (b) Let

$$f_1g_1 + f_2g_2 + \cdots + f_kg_k = 1$$

   where $f_i = \prod_{j \neq i} p_j^{r_j}$. Prove that $W_1 = f_1g_1(T)(V)$.

   (c) Prove that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

   (d) Let $T_1 = T_{|W_1}$ be the restriction of T. Prove that MMP of T_1 is $p_1^{r_1}$.