Math 790	**Test 3 (Takehome)**	Satya Mandal
Fall 05	Each Problem 10 points	Due on: October 2, 2005

Unless otherwise stated, F is a field and V, W are two vector sapces over F. .

1. Let V, W be two vector spaces over F and let $T : V \to W$ be a set theoretic map. Prove that the following are equivalent:

   (a) For $u, v \in V$ and $c, d \in F$ we have
   $$T(cu + dv) = cT(u) + dT(v)$$
   in W.

   (b) For $u, v \in V$ and $c \in F$ we have
   $$T(u+v) = T(u)+T(v) \quad \text{and} \quad T(cu) = cT(u)$$
   in W.

   (c) For $u, v \in V$ and $c \in F$ we have
   $$T(cu + v) = cT(u) + T(v)$$
   in W.

   (*Recall, T is said to be a linear transformation if one of (or all) the above conditions are satisfied.*)

2. Let V, W be two vector spaces over F. Let $e_1, e_2, \ldots, e_n$ be a basis of V and $w_1, w_2, \ldots, w_n \in$

W be n elments in W. Prove that there is EXACTLY one linear transformation

$$T : V \to W$$

such that

$$T(e_1) = w_1, T(e_2) = w_2, \ldots, T(e_n) = w_n.$$

3. Let V, W be two vector spaces over F and let $T : V \to W$ be a linear transformation. Assume $\dim(V) = n$ is finite. Prove that

$$rank(T) + nullity(T) = \dim(V).$$

4. Let A be an $m \times n$ matrix with entries in F. Prove that

$$row\ rank(A) = column\ rank(A).$$

5. Let V, W be two vector spaces over F and let $T : V \to W$ be a linear transformation. Assume that $\dim(V) = \dim(W) = n$ is finite. Prove that the following statements are equivalent:

   (a) T is invertible.

   (b) If $e_1, e_2, \ldots, e_m \in V$ (here $m \leq n$,) are linearly independent in V then the images

$T(e_1), T(e_2), \ldots, T(e_m)$ are linearly independent in W.

(c) T is onto.

6. Give the examples as follows:

   (a) Give an example of a linear operator $T : V \to V$ such that $T^2 = 0$ but $T \neq 0$.

   (b) Give two linear operator $T, U : V \to V$ such that $TU = 0$ but $UT \neq 0$.

7. Let V ba vector space and $T : V \to V$ be a linear operator. Assume that $rank(T) = rank(T^2)$. Prove that

$$range(T) \cap (Null\ Space(T)) = \{0\}.$$

8. Let V, W be two finite dimensional vector spaces over F. Assume $\dim V = n$ and $\dim W = m$. Let $M_{m,n}$ be the set of all $m \times n$ matrices with entries in F. Let $E = \{e_1, e_2, \ldots, e_n\}$ be a basis of V and $E' = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ be a basis of W.

   (a) For a linear transformation $T : V \to W$ define the matrix of T with respect to E and E'.

(b) Prove that the map

$$f : L(V, W) \to M_{m,n}$$

such that

$$f(T) = matrix\, of\, T\, with\, respect\, to\, E\, and\, E'$$

is an isomorphism.

(Try to understand the following diagram. Here A is the matrix of T.)

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow{\scriptstyle iso} & & \downarrow{\scriptstyle iso} \\ F^n & \xrightarrow{A} & F^m \end{array}$$

9. Let V be a finite dimensional vector space over F with $\dim(V) = n$ and

$$f : L(V, V) \to M_{n,n}$$

be the above isomorphism, with respect to a (same) fixed basis E. Prove that

(a) $f(TU) = f(T)f(U)$;

(b) $f(Id) = I_n$, the identity matrix;

(c) $T \in L(V, V)$ is an isomorphism if and only if $f(T)$ is an invertible matrix.

10. Let V be a finite dimensional vector space over F with $\dim(V) = n$. Let $E = \{e_1, \ldots, e_n\}$ and $E' = \{\epsilon_1, \ldots, \epsilon_n\}$ be two basis of V. Let $T \in L(V, V)$ be linear operator. Let

$$(e_1, \ldots, e_n) = (\epsilon_1, \ldots, \epsilon_n)P$$

for some $n \times n$ matrix.

(a) Prove that P is an invertible matrix.

(b) Let A be the matrix of T with respect to E and B be the matrix of T with respect to E'. Prove that $B = PAP^{-1}$.

11. Let V be a finite dimensional vector space over F with $\dim(V) = n$. Let $e_1, \ldots, e_n$ be a basis of V.

(a) Define the dual basis of $e_1, \ldots, e_n$. Also give a proof that it is indeed a basis of V^*.

(b) Let $W \subseteq V$ be subspace of V. Define the annihilator W^0 of W. Also prove that

$$\dim(W) + \dim(W^0) = n.$$

(c) For two subspaces W_1, W_2 of V prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

(d) For two subspaces W_1, W_2 of V prove that

$$(W_1 + W_2)^0 = W_1^0 \cap W_2^0$$

and

$$(W_1 \cap W_2)^0 = W_1^0 + W_2^0.$$