Determinants : Linear Algebra Notes

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1 Generalities

We will work with determinants of matrices over a commutative rings. Before we do that, we will want to talk about a modules over a commutative ring. A module over a ring is what vector spaces are over a filed.

Definition 1.1 *Let K be a commutative ring. A nonempty set M is said to be module over K if*

1. *M is an abelian group under addition $+$.*

2. *There is a scalar multiplication $K \times M \to M$. That means given $a \in K$ and $x \in M$ there is an element $ax \in M$.*

3. *Scalar multiplication is associative and distributive. That means for $a, b \in K$ and $x, y \in M$, we have (1) $(ab)x = a(bx)$, (2) $a(x+y) = ax + ay$, (3) $(a+b)x = ax + bx$. (4) $1x = x$*

Let K be a commutative ring and M be module over K.
Then

$$M^r = M \times M \times \ldots \times M$$

will denote cartesian product of $r-$copies M

Remark 1.1 Let K be a commutative ring and $V = K^n$. Therefore V is a $K-$module.

For an integer $n \geq 1$, let $\mathbb{M}_{mn}(K)$, or simply $\mathbb{M}_{mn}$ denote the group of all $m \times n$ matrices with coefficients in K. We also write $\mathbb{M}_n = \mathbb{M}_n(K) = \mathbb{M}_{nn}(K)$.

Let $A = (a_{ij})$ be an $n \times n$ matrix. Let $v_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ be the i^{th} row of A.

Note that

$$\begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{pmatrix} = A$$

is an element of V^n. (*We may write* $A = (v_1, \ldots, v_n)$ *as a row instead of a column, if conventinet.*)

In other words,

$$V^n = \mathbb{M}_{nn}.$$

Therefore, from whatever you know about determinants we have

$$det : V^n \to \mathbb{F}$$

is a function with various properties. We will formalize these propeties of determinats in the following definitions.

Definition 1.2 Let K be a commutative ring and M be a module over K. Let

$$D : M^r \to K$$

be a function.

1. D is said to be a **multilinear function** if D is linear in each coordinate. That means

$$D(x_1, \ldots, x_{i-1}, au + bv, x_{i+1}, \ldots, x_r) =$$

$$aD(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_r) + bD(x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_r)$$

   for all $a, b \in K$ and $x_i, u, v \in M$.

2. D is said to be an **alternating function** if (1) $D(x) = -D(y)$ where $x = (x_1, \ldots, x_r) \in M^r$ and y is found from x by switching x_i and x_j. (2) and $D(x) = 0$ whenever $x = (x_1, \ldots, x_r) \in M^r$ and $x_i = x_j$ for some $i \neq j$.

Lemma 1.1 *Let M be a module over a commutative ring K, and $2 \neq 0$ and $1/2 \in K$. Then for a function $D : M^r \to K$ the following are equivalent:*

1. *D is alternating.*
2. *$D(x) = -D(y)$ where $x = (x_1, \ldots, x_r) \in M^r$ and y is found from x by switching x_i and x_j.*

Proof. ((1) $\Rightarrow$ (2):) Obvious.

((2) $\Rightarrow$ (1):) Supose $x \in M^r$ is such that $x_i = x_j$ for some $i \neq j$. Then by $D(x) = -D(x)$. Therefore $2D(x) = 0$. Since $2 \neq 0$, we have 2 is invertible. So, $D(x) = 0$ and the proof is complete.

Definition 1.3 *Let K be a commutative ring and $\mathbb{M}_n(K)$ be the ring of all $n \times n$matrices. Note that $\mathbb{M}_n(K) = M^n$ where $M = K^n$. We say that a function*

$$D : M^n \to K$$

is a **determinant function** *if D is alternating, multilinear and also $D(I_n) = 1$.*

Remark 1.2 Obviously, (1) Multilinear corresponds to the fact that determinant are linear with respect to each row and (2) alternating corresponds to the property that when you switch two rows the determinant changes sign.

Theorem 1.1 *Let K be a commutative ring. Then there is a UNIQUE determinat function $D : \mathbb{M}_n(K) \to K$.*

We will denote this function by $det(*)$, or for a matrix $A \in \mathbb{M}_n(K)$ we will **denote** $D(A)$ **by** $\det(A)$.

Proof. Use induction on n.

2 Usual Facts

Theorem 2.1 Let K be a field.

1. For $A, B \in \mathbb{M}_n(K)$ we have $det(AB) = \det(A)\det(B)$.

2. For $A, B \in \mathbb{M}_n(K)$ if A is equivalent to B (i.e. $A = P^{-1}BP$ for some $P \in GL_n(\mathbb{F})$ also see page 94) then

$$det(A) = \det(A).$$

3. For $A \in \mathbb{M}_n(K)$ we have $det(A) = \det(A^t)$.

4. For square matrices $X, Y, W, Z \in \mathbb{M}_n(K)$ and

$$A = \begin{pmatrix} X & Z \\ W & Y \end{pmatrix}$$

we have

$$\det(A) = \det(X)\det(Y) - \det(Z)\det(W).$$

5. For square matrices $X \in \mathbb{M}_r(K), Y, \in \mathbb{M}_s(K)$ and a $r \times s$ matrix Z and

$$A = \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}$$

we have

$$\det(A) = \det(X)\det(Y).$$

6. For $A \in \mathbb{M}_n(K)$, denote the adjoint of A by $adj(A)$. Then

$$Aadj(A) = adj(A)A = \det(A)I_n.$$

7. bf (Permutations and determinants) Suppose S_n denote the group of permutations of the set $\{1, 2, \ldots, n\}$. For $\sigma \in S_n$, define $sign(\sigma) = 1$ if σ is even, else define $sign(\sigma) = -1$

   Now for $A \in \mathbb{M}_n(K)$ we have

$$\det(A) = \sum_{\sigma \in S_n} sign(\sigma)(\prod_{i=1}^{n} a_{i\sigma(i)})$$

Proof. Verify that the right hand side satisfies the definition (1.3) of the determinant function.

3 Determinant of Linear Operators

Let V be a vector space over a field $\mathbb{F}$ and $\dim(V) = n$. Let $L : V \to V$ be a linear operator.

Let A be the matrix of T with respect to a basis $e_1, \ldots, e_n$. The define

$$\det(T) := \det(A).$$

Note that by 2 of theorem 2.1, it follows that $\det(T)$ is well defined (i.e. does not depend on the basis).