Math 790 **Final** Satya Mandal

Fall 05 Each Problem 10 points December 15, 2005

Unless otherwise stated, $\mathbb{F}$ is a field.

1. Let A, B, C be square matrices.
   (a) Suppose B is a left inverse of A and C is right inverse of A. Prove that $B = C$.
   (b) Prove that inverse of a matrix, if it exists, is unique.
   (c) Suppose all these matrices are invertible. Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

2. Let V be a vector space over $\mathbb{F}$ and W_1, W_2 be two subspaces of V. Assume $W_1 + W_2$ has finite dimension. Prove that
$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

3. Let V, W be two vector spaces over $\mathbb{F}$ and let $T : V \to W$ be a linear transformation. Assume $\dim(V) = n$ is finite. Prove that
$$rank(T) + nullity(T) = \dim(V).$$

4. Suppose V is vector space of finite dimension, $\dim V = n$, over $\mathbb{F}$. Define the map
$$\varphi : L(V, V) \to L(V^*, V^*)$$
by $\varphi(T) = T^t$. Prove that φ is an isomorphism.

5. Suppose I is a non-zero ideal in $\mathbb{F}[X]$. Prove that $I = \mathbb{F}[X]d$ for some $d \in I$.

6. Suppose $T \in L(V, V)$ is a linear operator and $c_1, c_2, \ldots, c_k \in \mathbb{F}$ are the distinct eigen values of T. For $i = 1, \ldots, k$, let $W_i = N(c_i)$ be the eigen space of c_i.

   Prove that T is diagonalizable if and only if
$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

7. Suppose V is vector space over $\mathbb{F}$ with finite $\dim(V) = n$. Let $T \in L(V,V)$ be a linear operator. Suppose W is a $T-$invariant subspace of W and $T' = T_{|W}$ be the restriction.

   (a) Let q be the characteristic polynomial of T and Q be the characteristic polynomial of T'. Prove that $Q \mid q$.

   (b) Likewise, let p bet the MMP of T and P be the MMP of T'. Prove that $P \mid p$.

8. Let V be a finite dimensional vector space over a field $\mathbb{F}$. Suppose $E_1, \ldots, E_k$ are k linear operators on V satisfying all the conditions:

   (a) $E_i E_j = 0 \quad \forall \quad i \neq j$.

   (b) $E_1 + E_2 + \cdots + E_k = I$.

   Write $W_i = E_i(V)$. Prove that E_i is a projection and

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

9. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V,V)$ is an operator on V. Suppose $w_1, \ldots, w_r$ be such that

   (a) $V = \mathbb{F}[T]w_1 \oplus \mathbb{F}[T]w_2 \oplus \cdots \oplus \mathbb{F}[T]w_r$.

   (b) Let p_i be the MMP of w_i. Assume that $p_k \mid p_{k-1}$ for $k = 2, \ldots, r$.

   Prove $ann(T) = ann(w_1) = \mathbb{F}[X]p_1$.

10. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let V be an finite dimensional inner product space over $\mathbb{F}$. Let $T, U \in L(V,V)$ be two linear operator and $c \in \mathbb{F}$. Then

    (a) $(T+U)^* = T^* + U^*$,

    (b) $(cT)^* = \overline{c}T^*$,

    (c) $(TU)^* = U^* T^*$,

    (d) $(T^*)^* = T$.