Math 790	**Test 1 (Solutions)**	Satya Mandal
Fall 05	Each Problem 10 points	Due on: Spet 7, 2005

Unless otherwise stated, F is a field and all matrices have entries in F.

1. Supppose A is an $m \times n$ matrix with $m < n$. Prove that the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

2. Supppose A is a square matrix. Prove that A is row equivalent to the identity matrix if and only if $AX = 0$ has no non-trivial solution.

3. Write down the following elementary matrices :

   (*Note : An elmentary matrix is a square matrix. The textbook has a typo in the definition of elementary matrices* (page 20).)

   (a) The elementary matrix of size 4×4 that correspond to interchanging $2nd$ and $4th$ row.

   (b) The elementary matrix of size 4×4 that correspond to addition of $c - times$ the $2nd$ row to the $4th$ row.

   (c) The elementary matrix of size 4×4 that correspond to addition of $c - times$ the $4th$ row to the $2nd$ row.

   (d) The elementary matrix of size 4×4 that correspond to multiplying the $3rd$ row by scalar $c \neq 0$.

4. Let A, B, C be square matrices.

   (a) Suppose B is a left inverse of A and C is right inverse of A. Prove that $B = C$.

   (b) Prove that inverse of a matrix, if it exists, is unique.

   (c) Suppose all these matrices are invertible. Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Solution: Proof of (a): We have $B = BI = B(AC) = (BA)C = IC = C$. So, (a) is established.

Proof of (b): Suppose B, C be two inverses of A. So, B is a left inverse and C is a right inverse. Therefore, by (a), $B = C$. So, (b) is established.

Proof of (c): We have, $(ABC)(C^{-1}B^{-1}A^{-1}) = I = (C^{-1}B^{-1}A^{-1})(ABC)$. So, (c) is established.

5. Let A be a square matrix. Prove that the following are equivalent:

   (a) A is invertible,

   (b) A is row equivalent to the identity matrix,

   (c) A is product of elementary matrices,

   (d) the homogeneous system $AX = 0$ has only trivial solution,

   (e) for any constant vector Y (of appropriate size) the system $AX = Y$ has a solution.

6. Let A, B be two $m \times n$ square matrix. Prove that the following are equivalent:

   (a) A is rwo equivalent to B.

   (b) $A = PB$ where P is product of elementary matrices.

   (c) $A = PB$ where P is an invertible matrix.

7. Let A be a square matrix. Prove that the following are equivalent:

   (a) A is invertible.

   (b) A has a left inverse.

   (c) A has a right inverse.

   Solution: $((a) \Rightarrow (b))$: and $((a) \Rightarrow (c))$: are Obvious.

   $((b) \Rightarrow (a))$: Write $A = PR$ where P is invertible and R is R-R-E matrix. Let B be a left inverse of A. Therefore $BA = I$ and $BPR = I$. It follows that R has no zero row. Hence $R = I$ and $A = P$ is invertible.

   $((c) \Rightarrow (a))$: Now let C be right inverse of A. So, $AC = I$ and C has a left inverse. Since, we have proved that $(a) \Leftrightarrow (b)$, it follows that C has an inverse and $A = C^{-1}$.

8. Let A be an upper triangular matrix. Prove that A is invertible if and only if the each diagonal entry is non-zero.