Math 790	**Test 2 (Solutions)**	Satya Mandal
Fall 05	Each Problem 10 points	Due on: Spet 16, 2005

Unless otherwise stated, $\mathbb{F}$ is a field.

1. Let V be a vector space over $\mathbb{F}$ and W be a non-empty subset of V. Prove that the following are equivalent:

   (a) W is a subspace of V.

   (b) For $u, v \in W$ and $c, d \in \mathbb{F}$ we have $cu + dv \in W$.

   (c) For $u, v \in W$ and $c \in \mathbb{F}$ we have $u + v \in W$ and $cu \in W$.

   (d) For $u, v \in W$ and $c \in \mathbb{F}$ we have $cu + v \in W$.

   Solution. Similar to the proof of Problem-1 in Test-3.

2. Let V be a vector space over $\mathbb{F}$ and S be a non-emptysubset of V.

   (a) Define the subspace spanned by S. Write $W = Span(S)$.

   (b) Prove that if U is a subspace of V containing S, then W is contained in U.

   (c) Prove

$$W = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n : n \geq 0, c_i \in \mathbb{F}, v_i \in S\}.$$

Solution. (a) We define $Span(S)$ to be the intersection of all subspaces L of V that contain S. Notationally,

$$W = Span(S) = \cap\{L : L \ subspace \ \ of \ \ V, S \subseteq L\}.$$

Proof of (b) Suppose U is a subspace of V and $S \subseteq U$. Then, U is a member of the family on the RHS of the definition. So, $W \subseteq U$.

Proof of (c) Write

$$L_0 = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n : n \geq 0, c_i \in \mathbb{F}, v_i \in S\}.$$

Now, L_0 is a subspace of V and $S \subseteq L_0$. So, by (b), the span $W \subseteq L_0$.

Now suppose $v \in W$. Then $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ with $c_i \in \mathbb{F}$ and $v_i \in S$. If L is a subspace of V and $S \subseteq L$ then $v \in L$. Therefore, $v \in L$, for each menber L of the famlily of subspaces in the definition. So, $v \in \cap\{L : L \ subspace \ \ of \ \ V, S \subseteq L\} = W$. Therefore $L_0 \subseteq W$. So, the proof is complete.

3. Let V be a vector space over $\mathbb{F}$ and V is spanned by a finite set $S = \{v_1, \ldots, v_n\}$. Prove that a subset of S will form a basis of V.

Solution. If $V = \{0\}$, then empty set Φ forms a basis and the stament hold. So, assume $V \neq \{0\}$.

Let $i_1 = minimum\{i : v_{i_1} \neq 0\}$. Write $W_1 = Span(v_{i_1})$. Then v_{i_1} is linearly independent. If $W_1 = Span(v_{i_1}) = V$ then v_{i_1} is a basis of V and we are through.

So, we assume $W_1 \not\subseteq V$. Write $S_2 = \{v_i : i_1 < i \leq n\}$. Since $W_1 \not\subseteq V$ there are elements $v_i \in S_2 \setminus W_1$. Let $i_2 = minimum\{i : v_i \in S_2, v_i \notin W_1\}$.

Since $v_{i_2} \notin W_1$, we have v_{i_1}, v_{i_2} are linearly independent.

Write $W_2 = Span(v_{i_1}, v_{i_2})$. If $W_2 = V$ then v_{i_1}, v_{i_2} is a basis of V and we are through.

So, we assume $W_2 \not\subseteq V$. Since S is finite, this process must terminate and we will get a basis $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ of V. So, the proof is complete.

4. Let V be a finite dimensional vector space over $\mathbb{F}$ let $S = \{v_1, \ldots, v_n\}$ be a linearly independent subset. Prove that S extends to a basis of V. (*We really do not need to assume that V has finite dimension.*)

5. Let V be a vector space over $\mathbb{F}$ and V is spanned by a finite set $S = \{v_1, \ldots, v_n\}$. Prove that any two basis of V have same number of elements. (*We really do not need to assume that S is a finite set.*)

   Solution. Since V is spanned by a finite set, it has a finite basis. Suppose $e_1, \ldots, e_r$ be a basis of V with r elements and $E_1, \ldots, E_s$ be a basis of V with r elements.

   Suppose $r \neq s$. Assume $s < r$. We have

   $$(e_1, \ldots, e_r) = (E_1, \ldots, E_s)A$$

   for some $s \times r$ matrix A.

   Now the homgeneous system $AX = 0$ has s equations in r unknown. Since $s < r$, this system has a non-zero solution $C = (c_1, \ldots, c_r)^t$, where $c_i \in \mathbb{F}$. So, $AC^t = 0$. Therefore, $c_1e_1 + \cdots + c_re_r = 0$. This is contrdicts that $e_1, \ldots, e_r$ is a basis.

   Therefore $r = s$ and the proof is complete.

6. Let V be a vector space over $\mathbb{F}$ and W_1, W_2 be two subspaces of V. Assume $W_1 + W_2$ has finite dimension. Prove that

   $$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

7. Let A, B be two $m \times n$ matrices with entries in $\mathbb{F}$. Prove that A and B have same row space if and only if they are row equivalent.

8. Let $V = \mathbb{F}[X]$ be set of all polynomials over $\mathbb{F}$. Prove that, as a vector space, V does not have finite dimension.

   Solution. Suppose $\dim V = n$ is finite and $f_1, f_2, \ldots, f_n$ be basis of V. Let

   $$d = maximum\{degree(f_1), degree(f_2), \ldots, degree(f_n)\}.$$

   Then

   $$V = Span(f_1, f_2, \ldots, f_n) \subseteq \sum_{i=0}^{d} \mathbb{F}X^i.$$

   This is a contradiction. Because a polynomial f with $degree(f) > d$ is not in the sum on the right hand side. In particuler, $X^{d+1} \notin \sum_{i=0}^{d} \mathbb{F}X^i$. So, the proof is complete.