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| Math 790 | **Test 3 (Solutions)** | Satya Mandal |
| Fall 05 | Each Problem 10 points | Due on: October 2, 2005 |

**I like short proofs and elmentary proof.** Unless otherwise stated, $\mathbb{F}$ is a field and $V, W$ are two vector sapces over $\mathbb{F}$.

1. Let $V, W$ be two vector spaces over $\mathbb{F}$ and let $T : V \to W$ be a set theoretic map. Prove that the following are equivalent:

   (a) For $u, v \in V$ and $c, d \in \mathbb{F}$ we have
   $$T(cu + dv) = cT(u) + dT(v)$$
   in $W$.

   (b) For $u, v \in V$ and $c \in \mathbb{F}$ we have
   $$T(u + v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$
   in $W$.

   (c) For $u, v \in V$ and $c \in \mathbb{F}$ we have
   $$T(cu + v) = cT(u) + T(v)$$
   in $W$.

   (*Recall, $T$ is said to be a linear transformation if one of (or all) the above conditions are satisfied.*)

   **Solution.** ((a) $\Rightarrow$(b)): We have $T(cu + dv) = cT(u) + dT(v)$. Take $c = d = 1$, we get $T(u + v) = T(u) + T(v)$. Now take $d = 0$, we get $T(cu) = cT(u)$. Therefore (b) is established.

   ((b) $\Rightarrow$(c)): Using the additive part of the hypothesis, we have $T(cu + v) = T(cu) + T(v)$. Using $T(cu) = cT(u)$, we get $T(cu + v) = T(cu) + T(v) = cT(u) + T(v)$. Hence, (c) is established.

((c) $\Rightarrow$(a)): From the hypothesis in (c), we have $T(cu + dv) = cT(u) + T(dv)$. Also taking $v = 0$ we get $T(cu) = cT(u)$ for any $c \in \mathbb{F}$ and $u \in V$. So, we have

$$T(cu + dv) = cT(u) + T(dv) = cT(u) + dT(v)$$

and (a) is established.

2. Let $V, W$ be two vector spaces over $\mathbb{F}$. Let $e_1, e_2, \ldots, e_n$ be a basis of $V$ and $w_1, w_2, \ldots, w_n \in W$ be $n$ elments in $W$. Prove that there is EXACTLY one linear transformation

$$T : V \to W$$

such that

$$T(e_1) = w_1, T(e_2) = w_2, \ldots, T(e_n) = w_n.$$

3. Let $V, W$ be two vector spaces over $\mathbb{F}$ and let $T : V \to W$ be a linear transformation. Assume $\dim(V) = n$ is finite. Prove that

$$rank(T) + nullity(T) = \dim(V).$$

4. Let $A$ be an $m \times n$ matrix with entries in $\mathbb{F}$. Prove that

$$row\ rank(A) = column\ rank(A).$$

5. Let $V, W$ be two vector spaces over $\mathbb{F}$ and let $T : V \to W$ be a linear transformation. Assume that $\dim(V) = \dim(W) = n$ is finite. Prove that the following statements are equivalent:

   (a) $T$ is invertible.

   (b) If $e_1, e_2, \dots, e_m \in V$ (here $m \leq n$,) are linearly independent in $V$ then the images $T(e_1), T(e_2), \dots, T(e_m)$ are linearly independent in $W$.

   (c) $T$ is onto.

**Solution.** ((a) $\Rightarrow$(b)): Suppose $e_1, e_2, \dots, e_m \in V$ are linearly independent. We will prove that $T(e_1), T(e_2), \dots, T(e_m)$ are linearly independent. Suppose

$$c_1 T(e_1) + c_2 T(e_2) + \cdots + c_m T(e_m) = 0$$

for some $c_i \in \mathbb{F}$. Since $T$ is linear, we have $T(c_1 e_1 + \cdots + c_m e_m) = c_1 T(e_1) + c_2 T(e_2) + \cdots + c_m T(e_m) = 0$. By (a), $T$ is invertible and hence one to one. Therefore $c_1 e_1 + \cdots + c_m e_m = 0$. By linear indpendence of $e_1, \dots, e_m$, we have $c_i = 0$. Therefore (b) is established.

((b) $\Rightarrow$(c)): Suppose $e_1, e_2, \dots, e_n$ is a basis of $V$. By (b), $T(e_1), T(e_2), \dots, T(e_n)$ are linearly independent. Since $\dim(W) = n$, it follows that $T(e_1), T(e_2), \dots, T(e_n)$ is a basis of $W$. Therefore $T(V) =$

$$T(Span(e_1, \dots, e_n)) = Span(T(e_1), T(e_2), \dots, T(e_n)) = W.$$

Hence $T$ is onto and (c) is established.

((c) $\Rightarrow$(a)): We need to show, $T$ is one to one. We have $nullity(T) + rank(T) = n$. Since $T$ is onto, $rank(T) = n$. Hence, $nullity(T) = 0$. So, null space of $T$ is zero and $T(v) = 0$ implies $v = 0$. Hence $T$ is one to one. So, (a) is established.

6. Give the examples as follows:

   (a) Give an example of a linear operator $T : V \to V$ such that $T^2 = 0$ but $T \neq 0$.

   (b) Give two linear operator $T, U : V \to V$ such that $TU = 0$ but $UT \neq 0$.

7. Let $V$ ba vector space and $T : V \to V$ be a linear operator. Assume that $rank(T) = rank(T^2)$. Prove that

$$range(T) \cap (Null\ Space(T)) = \{0\}.$$

**Solution.** $N_T$ and $N_{T^2}$ will denote the null space of $T$ and $T^2$, respectively. First, note that $N_T \subseteq N_{T^2}$.

Since $\dim(N_T) + rank(T) = n = \dim(N_{T^2}) + rank(T^2)$, we have $\dim(N_T) = \dim(N_{T^2})$. Therefore, $N_T = N_{T^2}$.

Now suppose

$$x \in range(T) \cap (Null\ Space(T)).$$

So, $x = T(y)$ for some $y \in V$. Since $T(x) = 0$ we have $y \in N_{T^2} = N_T$. Therfore $x = T(y) = 0$ and the proof is complete.

8. Let $V, W$ be two finite dimensional vector spaces over $\mathbb{F}$. Assume $\dim V = n$ and $\dim W = m$. Let $M_{m,n}$ be the set of all $m \times n$ matrices with entries in $\mathbb{F}$. Let $E = \{e_1, e_2, \ldots, e_n\}$ be a basis of $V$ and $E' = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ be a basis of $W$.

   (a) For a linear transformation $T : V \to W$ define the matrix of $T$ with respect to $E$ and $E'$.

   (b) Prove that the map

   $$f : L(V, W) \to M_{m,n}$$

   such that

   $$f(T) = \textit{matrix of } T \textit{ with respect to } E \textit{ and } E'$$

   is an isomorphism.

   (*Try to understand the following diagram. Here $A$ is the matrix of $T$.*)

   $$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \Big\downarrow \scriptstyle{iso} & & \Big\downarrow \scriptstyle{iso} \\ \mathbb{F}^n & \xrightarrow{A} & \mathbb{F}^m \end{array}$$

**Solution.** We will prove only (b). *Let me comment that to prove that $f$ is 'isomorphism', there are two general methods. First method proves that the map $f$ is one to one and onto. Alternately, you can define a map $g$ in the opposite direction and prove that $fg = Id$ and $gf = Id$. I will write a proof in using this alternatve method.*

Define a map

$$g : M_{m,n} \to L(V, W)$$

as follows: For $A \in M_{m,n}$ define $T \in L(V, W)$ by the equation:

$$(T(e_1), \ldots, T(e_n)) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)A$$

and let $g(A) = T$.

Note $g$ is linear and $gf = Id_{L(V,W)}$ and $fg = Id_{M_{m,n}}$. So, $g$ is the inverse of $f$ and the proof is complete.

9. Let $V$ be a finite dimensional vector space over $\mathbb{F}$ with $\dim(V) = n$ and

$$f : L(V, V) \to M_{n,n}$$

be the above isomorphism, with respect to a (same) fixed basis $E$. Prove that

(a) $f(TU) = f(T)f(U)$;

(b) $f(Id) = I_n$, the identity matrix;

(c) $T \in L(V, V)$ is an isomorphism if and only if $f(T)$ is an invertible matrix.

**Solution.** Suppose $e_1, e_2, \ldots, e_n$ be a basis of $V$ and $f$ is defined with respect to this basis.

**Proof of (a):** Write $f(T) = A \in M_{n,n}$ and $f(U) = B \in M_{n,n}$. Then

$$(T(e_1), \ldots, T(e_n)) = (e_1, \ldots, e_n)A$$

and

$$(U(e_1), \ldots, U(e_n)) = (e_1, \ldots, e_n)B.$$

Apply $T$ to the second one and then use the first one. We get

$$(TU(e_1), \ldots, TU(e_n)) = (T(e_1), \ldots, T(e_n))B = (e_1, \ldots, e_n)AB.$$

So, the matrix of $TU$ is $AB$. Hence $f(TU) = AB = f(T)f(U)$ and the proof of (a) is complete.

**Proof of (b):** We have $(Id(e_1), \ldots, Id(e_n)) = (e_1, \ldots, e_n)I_n$. Therefore $f(Id) = I_n$.

**Proof of (c):** $(\Rightarrow)$: Suppose $T$ has a inverse $T^{-1}$. Then $TT^{-1} = T^{-1}T = Id$. Now use (a) and (b). We have

$f(TT^{-1}) = f(T^{-1}T) = f(Id) = I_n$. By (a) $f(T)f(T^{-1}) = f(T^{-1})f(T) = I_n$. Therefore $f(T^{-1})$ is the inverse of $f(T)$.

$(\Rightarrow)$: Write $f(T) = A$. Suppose $A$ is ivertible. Let $B$ be the inverse of $A$. Since $f$ is onto, $f(U) = B$ for some $U \in M_{n,n}$. So, $f(TU) = f(T)f(U) = AB = I_n$. Since $f$ is one to one, $TU = Id$. Similarly, $UT = Id$ and therefore, $T$ is invertible. This completes the proof.

10. Let $V$ be a finite dimensional vector space over $\mathbb{F}$ with $\dim(V) = n$. Let $E = \{e_1, \ldots, e_n\}$ and $E' = \{\epsilon_1, \ldots, \epsilon_n\}$ be two basis of $V$. Let $T \in L(V, V)$ be linear operator. Let

$$(e_1, \ldots, e_n) = (\epsilon_1, \ldots, \epsilon_n)P$$

for some $n \times n$ matrix.

(a) Prove that $P$ is an invertible matrix.

(b) Let $A$ be the matrix of $T$ with respect to $E$ and $B$ be the matrix of $T$ with respect to $E'$. Prove that $B = PAP^{-1}$.

**Solution. Proof of (a):** There is also a matrix $Q$ such that

$$(\epsilon_1, \ldots, \epsilon_n) = (e_1, \ldots, e_n)Q.$$

Combining these two, we get

$$(e_1, \ldots, e_n) = (e_1, \ldots, e_n)QP.$$

Therefore $QP = I_n$ and similarly, $PQ = I_n$. So, $P$ is invertible.

**Proof of (b):** Apply $T$ to the equation:

$$(e_1, \ldots, e_n) = (\epsilon_1, \ldots, \epsilon_n)P.$$

We get

$$(T(e_1), \ldots, T(e_n)) = (T(\epsilon_1), \ldots, T(\epsilon_n))P.$$

We also have

$$(T(e_1), \ldots, T(e_n)) = (e_1, \ldots, e_n)A$$

and

$$(T(\epsilon_1), \ldots, T(\epsilon_n)) = (\epsilon_1, \ldots, \epsilon_n)B.$$

Therefore

$$(e_1, \ldots, e_n)A = (\epsilon_1, \ldots, \epsilon_n)BP = (e_1, \ldots, e_n)P^{-1}BP.$$

Comparing, we get $A = P^{-1}BP$. This completes the proof.

11. Let $V$ be a finite dimensional vector space over $\mathbb{F}$ with $\dim(V) = n$. Let $e_1, \ldots, e_n$ be a basis of $V$.

(a) Define the dual basis of $e_1, \ldots, e_n$. Also give a proof that it is indeed a basis of $V^*$.

(b) Let $W \subseteq V$ be subspace of $V$. Define the annihilator $W^0$ of $W$. Also prove that

$$\dim(W) + \dim(W^0) = n.$$

(c) For two subspaces $W_1, W_2$ of $V$ prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

(d) For two subspaces $W_1, W_2$ of $V$ prove that

$$(W_1 + W_2)^0 = W_1^0 \cap W_2^0$$

and

$$(W_1 \cap W_2)^0 = W_1^0 + W_2^0.$$