Math 790 **Test 4 (Solutions)** Satya Mandal
Fall 05 Each Problem 10 points Due on: October 10, 2005

I like short proofs and direct proofs.

1. Suppose V is a vector space over $\mathbb{F}$ and $W \subseteq V$ is a subspace of V. Prove that annihilator of the the annihilator of W is itself. That is, notationally, prove that $W = W^{00}$.

2. Suppose V is vector space of finite dimension, $\dim V = n$, over $\mathbb{F}$. Let $g, f_1, \ldots, f_r \in V^*$ be linear functionals. Let N be the null space of g and N_i be the null space of f_i.

   Then, $N_1 \cap N_2 \cap \cdots \cap N_r \subseteq N$ if and only if $g = \sum_{i=1}^{r} c_i f_i$ for some $c_i \in \mathbb{F}$.

3. Suppose V is a vector space over $\mathbb{F}$ and $W \subseteq V$ is a subspace of V. Suppose $g_1, \ldots, g_r \in V^*$ forms a basis of the annihilator W^0. Write $N_i = Null(g_i)$. Prove that

$$W = \cap_{i=1}^{r} N_i.$$

Solution: Write $N = \cap_{i=1}^{r} N_i$. Suppose $w \in W$. Since $g_i \in W^0$ we have $g_i(w) = 0$. Therefore $w \in N_i$ for $i = 1, \ldots, r$. Hence

$$W \subseteq N.$$

Now suppose $W \neq N$. Then there is

$$e \in N \quad such \;\; that \quad e \notin W.$$

We will construct a functional $f \in W^0$ such that $f(e) \neq 0$. Suppose $e_1, \ldots, e_k$ is a basis of W. Write $e_{k+1} = e$. Note that $e_1, \ldots, e_k, e_{k+1}$

are linearly independent. Extend this to a basis
$e_1, \ldots, e_k, e_{k+1}, e_{k+2}, \ldots, e_n$ of V. Now define $f : V \to \mathbb{F}$ by $f(e_{k+1}) = 1$ and $f(e_i) = 0$ for $i = 1, \ldots, n; i \neq k+1$. (*That means* $f = e^*_{k+1}$.)
Then $f \in W^0$ and $f(e) \neq 0$. By hypothesis, $f = c_1 g_1 + \cdots + c_r g_r$. Since $e \in N$, we have $g_i(e) = 0$ for $i = 1, \ldots, r$. Therefore $f(e) = 0$, which is contradiction.

(*Sure, I like my proof. But I do not mean that this is the only way or the best way.*)

4. Suppose V, W be two finite dimensional vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation and $T^t : W^* \to V^*$ be the transpose. Prove that $rank(T) = rank(T^*)$.

   Also prove that for a $m \times n$ matrix A with entries in A, we have $row - rank(A) = column - rank(A)$.

5. Suppose V is vector space of finite dimension, $\dim V = n$, over $\mathbb{F}$. Define the map

   $$\varphi : L(V, V) \to L(V^*, V^*)$$

   by $\varphi(T) = T^t$. Prove that φ is an isomorphism.

   Solution: Note that

   $$\dim(L(V, V)) = \dim(L(V^*, V^*)) = n^2.$$

   So, it is enough to prove that φ is one to one. Suppose $T \in L(V, V)$ and $\varphi(T) = 0$, we will prove that $T = 0$. Now, we have $T^t = 0$. Suppose $e_1, \ldots, e_n$ is a basis of V and A is the matrix of T. Then A^t is the matrix of T^t with respect to the dual basis.

Since $T^t = 0$, we have $A^t = 0$. This implies $A = 0$ and hence $T = 0$. Hence φ is one to one and therefore an isomorphism.