Theorem 0.1 *Let V, W be two finite dimentional vector spaces over a field F with $\dim V = \dim W = n$. Suppose $T : V \to W$ is a linear transformation. Then the following are equivalent:*

1. *T is invertible.*
2. *T is nonsingular. (That means $(T(v) = 0) \Rightarrow (v = 0)$.)*
3. *T is onto.*
4. *If $e_1, e_2, \ldots, e_n$ is a basis of V then $T(e_1), T(e_2), \ldots, T(e_n)$ is a basis of W.*
5. *There is a basis $e_1, e_2, \ldots, e_n$ of V such that $T(e_1), T(e_2), \ldots, T(e_n)$ is a basis of W.*

Proof. $(1) \Rightarrow (2)$: Obvious.
$(2) \Rightarrow (3)$: Suppose $e_1, e_2, \ldots, e_n$ is a basis of V.
Claim that $T(e_1), T(e_2), \ldots, T(e_n)$ is a basis of W. Supppose $a_1T(e_1) + a_2T(e_2) + \cdots + a_nT(e_n) = 0$ for some $a_i \in F$. Then $T(a_1e_1 + a_2e_2 + \cdots + a_ne_n) = a_1T(e_1) + a_2T(e_2) + \cdots + a_nT(e_n) = 0$. Since T is nonsingular $a_1e_1 + a_2e_2 + \cdots + a_ne_n = 0$. Since $e_1, \ldots, e_n$ are linearly independent, $a_i = 0$ for $i = 1, 2, \ldots, n$. So, $T(e_1), T(e_2), \ldots, T(e_n)$ are linearly independent. Since, $\dim W = n$, indeed $T(e_1), T(e_2), \ldots, T(e_n)$ is a basis of W. Therefore $W \subseteq Span(T(e_1), T(e_2), \ldots, T(e_n)) = T(V)$. So, T is onto.
$(3) \Rightarrow (1)$: Let $w_1, \ldots, w_n$ be basis of W. Since T is onto, $T(e_i) = w_i$ for $i = 1, \ldots, n$ for some $e_i \in V$. It follows that $e_1, \ldots, e_n$ are linearly independent and hence a basis of V, because $\dim V = n$.

Now, suppose $T(v) = 0$ for some $v \in V$. Then $v = a_1e_1 + \cdots + a_ne_n$ for some $a_i \in F$. Apply T and get $0 = a_1T(e_1) + \cdots + a_nT(e_n) = a_1w_1 + \cdots + a_nw_n$. Therefore $a_i = 0$ for $i = 1, \ldots, n$ and hence $v = 0$. So, T is also nonsingular and hence T is an isomorphism.
$(3) \Rightarrow (4)$: Since T is onto, $W = Span\{T(e_1), T(e_2), \ldots, T(e_n)\}$. Since $\dim W = n$, indeed, $T(e_1), T(e_2), \ldots, T(e_n)$ is a basis of W and hence linearly independent.
$(4) \Rightarrow (5)$: Let $e_1, \ldots, e_n$ ne any basis of V. By (4), we have, $T(e_1), T(e_2), \ldots, T(e_n)$ is a basis of W.

(5) $\Rightarrow$ (3) : By (5), there is a basis $e_1, \ldots, e_n$ of V such that $T(e_1), T(e_2), \ldots, T(e_n)$ is a basis of W. Then, given any $w \in W$, we have $w = a_1 T(e_1) + \cdots + a_n T(e_n) = T(a_1 e_1 + \cdots + a_n e_n)$. Therefore $w \in T(W)$ and T is onto.